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Quasi-polynomial tractability of linear problems in the average case setting ☆

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ABSTRACT

We study d-variate approximation problems in the average case setting with respect to a zero-mean Gaussian measure. We consider algorithms that use finitely many evaluations of arbitrary linear functionals. For the absolute error criterion, we obtain the necessary and sufficient conditions in terms of the eigenvalues of its covariance operator and obtain an estimate of the exponent $t^{\text{qpol-avg}}$ of quasi-polynomial tractability which cannot be improved in general. For the linear tensor product problems, we find that the quasi-polynomial tractability is equivalent to the strong polynomial tractability. For the normalized error criterion, we solve a problem related to the Korobov kernels, which is left open in Lifshits et al. (2012).

1. Introduction

Multivariate computational problems are defined on classes of functions depending on d variables with large or even huge d. Multivariate problems occur in many applications such as in computational finance, statistics and physics. Such problems are usually solved by algorithms that use finitely many information operators. One information operator is defined as one function value or the evaluation of one linear functional. The minimal number of information operators needed to find the solution to within ε is the intrinsic difficulty of the problem. It is called the information complexity and is denoted by $n(\varepsilon, d)$ to stress the dependence on the two important parameters.

Research on tractability of multivariate continuous problems started in 1994 (see [12]). Tractability of multivariate problems studies when $n(\varepsilon, d)$ is not exponential in ε^{-1} and d. A problem is intractable if the information complexity is an exponential function of ε^{-1} or d. Otherwise, the problem is

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tractable. These problems have been studied for different error criteria and in different settings including the worst and average case setting. Different kinds of tractable problems have been considered in the literature. In fact, tractability of multivariate problems has been recently a very active research area: see [6–8] and the references therein. We list the different tractability concepts below.

- Weak tractability if $n(\varepsilon, d)$ is not exponential in d and ε^{-1}.
- Quasi-polynomial tractability if $n(\varepsilon, d)$ is of order $\exp(t(1 + \ln d)(1 + \ln \varepsilon^{-1}))$ (a special case of the T-tractability (see [1])).
- Polynomial tractability if $n(\varepsilon, d)$ is of order $d^q \varepsilon^{-p}$.
- Strong polynomial tractability if $n(\varepsilon, d)$ is of order ε^{-p}.

The bounds above hold for all $d \in \mathbb{N}$ and all $\varepsilon \in (0, 1)$ with the parameters t, q, p and the pre-factors independent of d and ε.

The concept of quasi-polynomial tractability has been introduced recently in [2]. Similar investigations can consult in [3–5]. The main purpose of this paper is to study the quasi-polynomial tractability of linear problems in the average case setting, and this is done for the class of arbitrary linear functionals Λ^{all}.

In Section 2, we study the quasi-polynomial tractability of general linear multivariate problems defined over Hilbert spaces. For the absolute error criterion, we find the necessary and sufficient conditions in terms of the eigenvalues of its covariance operator. Besides, we obtain an estimate of the exponent $t^{\text{qpol-avg}}$ of quasi-polynomial tractability which cannot be improved in general.

In Section 3, we study the quasi-polynomial tractability of linear tensor product problems $S = \{S_d\}$ defined over Hilbert spaces. We find that for the absolute error criterion, S is quasi-polynomially tractable iff S is strongly polynomially tractable.

In Section 4, we study the quasi-polynomial tractability of a multivariate approximation problem whose covariance kernel is given as a Korobov kernel. For the normalized error criterion, we solve a problem which is left open in [4].

2. Linear problems defined over Hilbert spaces

In this section we consider multivariate problems in the average case setting for the absolute error criterion. First, we recall multivariate problems in the average case setting (see [11]).

Let F_d be a Banach space of d-variate real functions defined on a Lebesgue measurable set $D_d \subset \mathbb{R}^d$. The space F_d is equipped with a zero-mean Gaussian measure μ_d defined on Borel sets of F_d. We denote by $C_{\mu_d} : F_d^* \to F_d$ (where and in the following F_d^* denotes the dual space of F_d) the covariance operator of μ_d, e.g., see [6, Appendix B] for its definition. Let H_d be a Hilbert space with inner product and norm denoted by $\langle \cdot, \cdot \rangle_{H_d}$ and $\| \cdot \|_{H_d}$, respectively.

We want to approximate a continuous linear operator

$$S_d : F_d \to H_d.$$

Let $\nu_d = \mu_d S_d^{-1}$ be the induced measure. Then ν_d is a zero-mean Gaussian measure on the Borel sets of H_d with covariance operator $C_{\nu_d} : H_d \to H_d$ given by

$$C_{\nu_d} = S_d C_{\mu_d} S_d^*,$$

where $S_d^* : H_d \to F_d^*$ is the operator dual to S_d. Then C_{ν_d} is self-adjoint, nonnegative definite, and has finite trace. Let $(\lambda_{d,j}, \eta_{d,j})_{j=1,2,\ldots}$ denote its eigenpairs

$$C_{\nu_d} \eta_{d,j} = \lambda_{d,j} \eta_{d,j} \quad \text{with } \lambda_{d,1} \ge \lambda_{d,2} \ge \cdots.$$

Then

$$\operatorname{trace}(C_{\nu_d}) = \sum_{j=1}^{\infty} \lambda_{d,j} = \int_{H_d} \|g\|_{H_d}^2 \nu_d(dg) = \int_{F_d} \|S_d f\|_{H_d}^2 \mu_d(df) < \infty.$$