Journal of Complexity 30 (2014) 69–86

Contents lists available at ScienceDirect

Journal of Complexity

journal homepage: www.elsevier.com/locate/jco

Weighted discrepancy and numerical integration in function spaces

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ARTICLE INFO

Article history:
Received 24 May 2013
Accepted 13 August 2013
Available online 30 August 2013

Keywords:
Discrepancy
Numerical integration
Faber bases
Besov–Sobolev spaces
Dominating mixed smoothness

ABSTRACT

The paper deals with weighted discrepancy and numerical integration in Euclidean n-space in the context of Faber bases for Besov–Sobolev spaces with dominating mixed smoothness.

1. Introduction and main assertions

Let $Q = (0,1)^n$ be the unit cube in $\mathbb{R}^n$, $n \in \mathbb{N}$, and $Q_M = M + Q$ with $M = (M_1, \ldots, M_n) \in \mathbb{Z}^n$. Let $\Gamma = \{x^j\}_{j=1}^k$ be a set of $k \in \mathbb{N}$ points $x^j = (x^j_1, \ldots, x^j_n)$ in $\mathbb{R}^n$ and

$$R^j_\Gamma = \left\{ x \in Q_M : x^j_l < x_l < M_l + 1,\ l = 1, \ldots, n \right\} \quad \text{if } M_l \le x^j_l < M_l + 1,$$

$j = 1, \ldots, k$, be rectangles anchored at the upper right corner of related cubes Q_M. Let $\chi_{R^j_\Gamma}$ be the characteristic function of R^j_Γ and let $A = \{a_j\}_{j=1}^k \subset \mathbb{C}$. Then the discrepancy function

$$\mathrm{disc}_{\Gamma,A}(x) = \prod_{l=1}^n (x_l - M_l) - \sum_{j: R^j_\Gamma \subset Q_M} a_j\, \chi_{R^j_\Gamma}(x) \quad \text{if } x \in Q_M, \tag{1.1}$$

$M \in \mathbb{Z}^n$, extends the well-known discrepancy function from Q to $\mathbb{R}^n$ modulo 1 as suggested in the classical papers [24,7]. If for given Q_M there are no R^j_Γ with $R^j_\Gamma \subset Q_M$ then $\mathrm{disc}_{\Gamma,A}(x) = \prod_{l=1}^n (x_l - M_l)$.

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0885-064X/$ – see front matter
http://dx.doi.org/10.1016/j.jco.2013.08.002

Furthermore $\mathrm{disc}_{\Gamma,A}(x) = 0$ for $x \in \mathbb{R}^n \setminus \bigcup_M Q_M$ (the faces of all Q_M). We measure $\mathrm{disc}_{\Gamma,A}$ in weighted function spaces. Let

$$w^\alpha(x) = \prod_{l=1}^{n} (1 + x_l^2)^{\alpha/2}, \quad x \in \mathbb{R}^n,\ \alpha \in \mathbb{R}. \tag{1.2}$$

We are mainly interested in weighted Besov spaces $S^r_{p,p}B(\mathbb{R}^n, \alpha)$ of dominating mixed smoothness and related weighted (fractional) Sobolev spaces $S^r_p H(\mathbb{R}^n, \alpha)$ of dominating mixed smoothness defined below. Recall that $S^0_p H(\mathbb{R}^n, \alpha) = L_p(\mathbb{R}^n, \alpha)$, $1 < p < \infty$, normed by

$$\| f \,| L_p(\mathbb{R}^n, \alpha)\| = \left(\int_{\mathbb{R}^n} w^\alpha(x)^p |f(x)|^p \, \mathrm{d}x \right)^{1/p}, \quad \alpha \in \mathbb{R}. \tag{1.3}$$

We extend the classical discrepancy $\mathrm{disc}_k \left(L_p(Q) \right)$, $1 < p < \infty$, $k \in \mathbb{N}$, and its modifications $\mathrm{disc}_k \left(S^r_{p,q} A(Q) \right)$, where $S^r_{p,q}A(Q)$ are suitable spaces with dominating mixed smoothness in Q as considered in [20], to some weighted spaces on $\mathbb{R}^n$. Then

$$\mathrm{disc}_k \left(S^r_{p,p}B(\mathbb{R}^n, \alpha) \right) = \inf \left\| \mathrm{disc}_{\Gamma,A} \,| S^r_{p,p}B(\mathbb{R}^n, \alpha) \right\| \tag{1.4}$$

where the infimum is taken over all $\Gamma = \{x^j\}_{j=1}^k \subset \mathbb{R}^n$ and all $A = \{a_j\}_{j=1}^k \subset \mathbb{C}$. Similarly for

$$\mathrm{disc}_k \left(S^r_p H(\mathbb{R}^n, \alpha) \right) = \inf \left\| \mathrm{disc}_{\Gamma,A} \,| S^r_p H(\mathbb{R}^n, \alpha) \right\|. \tag{1.5}$$

We add a comment below under which restrictions for r, p, α both (1.4) and (1.5) make sense. The close connection between discrepancy in $L_p(Q)$, $1 < p < \infty$, and numerical integration in $L_{p'}(Q)$ is one of the cornerstones of this theory. We extended this relation in [20] to some spaces $S^r_{p,q}A(Q)$. One may ask for weighted counterparts. Let UA be the unit ball in the Banach space A. Let

$$1 < p < \infty, \qquad \alpha + \frac{1}{p} > 1 \quad \text{and} \quad r > \frac{1}{p}. \tag{1.6}$$

Then

$$\mathrm{Int}_k \left(S^r_{p,p}B(\mathbb{R}^n, \alpha) \right) = \inf \left[\sup_{f \in US^r_{p,p}B(\mathbb{R}^n,\alpha)} \left| \int_{\mathbb{R}^n} f(x) \, \mathrm{d}x - \sum_{j=1}^k a_j f(x^j) \right| \right], \tag{1.7}$$

$k \in \mathbb{N}$, where the infimum is taken over all $\{x^j\}_{j=1}^k \subset \mathbb{R}^n$ and all $A = \{a_j\}_{j=1}^k \subset \mathbb{C}$. This is the extension of [21, Definition 4.11, p. 93] from $\mathbb{R}^2$ to $\mathbb{R}^n$ as indicated in [21, Section 4.5]. For discussions and justifications we refer to [20, Chapter 5] and [21]. We only mention that $r > 1/p$ ensures that pointwise evaluation $f(x^j)$ makes sense, which will also be discussed later on in connection with Faber bases. Furthermore it follows from $\alpha + \frac{1}{p} > 1$ that $S^r_{p,p}B(\mathbb{R}^n, \alpha) \hookrightarrow L_1(\mathbb{R}^n)$. Both together justifies (1.7). We need a minor modification of these integral numbers. Let $f^\urcorner$ for $f \in S^r_{p,p}B(\mathbb{R}^n, \alpha)$ be as in Remark 2.9 and Corollary 2.8 below. Then

$$\mathrm{Int}_k \left(S^r_{p,p}B(\mathbb{R}^n, \alpha)^\urcorner \right) = \inf \left[\sup_{f \in US^r_{p,p}B(\mathbb{R}^n,\alpha)} \left| \int_{\mathbb{R}^n} f^\urcorner(x) \, \mathrm{d}x - \sum_{j=1}^k a_j f^\urcorner(x^j) \right| \right], \tag{1.8}$$

$k \in \mathbb{N}$, where the infimum has the same meaning as above. It is the main aim of this paper to prove the following assertions. But first we fix our use of $\sim$ (equivalence) as follows. Let I be an arbitrary index set. Then $a_i \sim b_i$ for two sets of positive numbers $\{a_i : i \in I\}$ and $\{b_i : i \in I\}$ means that there are two positive numbers c_1 and c_2 such that $c_1 a_i \le b_i \le c_2 a_i$ for all $i \in I$.

Theorem 1.1. *Let $n \in \mathbb{N}$ and*

$$1 < p < \infty, \qquad \frac{1}{p} - 1 < r < \frac{1}{p}, \qquad \alpha < -\frac{1}{p}. \tag{1.9}$$