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On lower bounds for integration of multivariate permutation-invariant functions

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A B S T R A C T

In this note we study multivariate integration for permutation-invariant functions from a certain Banach space $E_{d,\alpha}$ of Korobov type in the worst case setting. We present a lower error bound which particularly implies that in dimension d every cubature rule which reduces the initial error necessarily uses at least $d + 1$ function values. Since this holds independently of the number of permutation-invariant coordinates, this shows that the integration problem can never be strongly polynomially tractable in this setting. Our assertions generalize results due to Sloan and Woźniakowski (1997) [3]. Moreover, for large smoothness parameters α our bound cannot be improved. Finally, we extend our results to the case of permutation-invariant functions from Korobov-type spaces equipped with product weights.

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1. Introduction and main result

Consider the integration problem $\mathrm{Int} = (\mathrm{Int}_d)_{d\in\mathbb{N}}$,

$$\mathrm{Int}_d : E_{d,\alpha} \to \mathbb{C}, \qquad \mathrm{Int}_d(f) = \int_{[0,1]^d} f(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x},$$

for periodic, complex-valued functions in the Korobov class

$$E_{d,\alpha} := \left\{ f \in L_1([0,1]^d) \mid \|f\| := \left\| f \mid E_{d,\alpha} \right\| := \sup_{\boldsymbol{k}\in\mathbb{Z}^d} \left|\widehat{f}(\boldsymbol{k})\right| \left(\overline{k_1} \cdot \ldots \cdot \overline{k_d}\right)^{\alpha} < \infty \right\}$$

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where $d \in \mathbb{N}$ and $\alpha > 1$. Here $\mathbb{Z}$ denotes the set of integers, $\mathbb{N} := \{1, 2, \ldots\}$, and we set $\overline{k_m} := \max\{1, |k_m|\}$. Moreover, for $f \in L_1([0,1]^d)$

$$\widehat{f}(\boldsymbol{k}) := \left\langle f, e^{2\pi \mathrm{i}\boldsymbol{k}\cdot}\right\rangle_{L_2} := \int_{[0,1]^d} f(\boldsymbol{x})\, e^{-2\pi \mathrm{i}\boldsymbol{k}\boldsymbol{x}}\, \mathrm{d}\boldsymbol{x}, \quad \boldsymbol{k} = (k_1, \ldots, k_d) \in \mathbb{Z}^d,$$

denotes its $\boldsymbol{k}$th Fourier coefficient, where $\boldsymbol{k}\boldsymbol{x} = \sum_{m=1}^{d} k_m \cdot x_m$, and $\mathrm{i} = \sqrt{-1}$. To approximate $\mathrm{Int}_d(f)$, without loss of generality, we consider algorithms from the class of all linear cubature rules

$$\mathcal{A}(f) := \mathcal{A}_{N,d}(f) := \sum_{n=1}^{N} w_n f\left(\boldsymbol{t}^{(n)}\right), \quad N \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}, \tag{1}$$

that use at most N values of the input function f at some points $\boldsymbol{t}^{(n)} \in [0,1]^d$, $n = 1, \ldots, N$. The weights w_n can be arbitrary complex numbers. Clearly, every function $f \in E_{d,\alpha}$ has a 1-periodic extension since their Fourier series are absolutely convergent:

$$\sum_{\boldsymbol{k}\in\mathbb{Z}^d} \left|\widehat{f}(\boldsymbol{k})\, e^{2\pi \mathrm{i}\boldsymbol{k}\cdot}\right| \leq \|f\| \cdot \sum_{\boldsymbol{k}\in\mathbb{Z}^d} \left(\overline{k_1} \cdot \ldots \cdot \overline{k_d}\right)^{-\alpha} = \|f\| \cdot (1 + 2\,\zeta(\alpha))^d < \infty.$$

As usual, $\zeta(s) = \sum_{m=1}^{\infty} m^{-s}$ is the Riemann zeta function evaluated at $s > 1$.

In [3] Sloan and Woźniakowski showed that for every $d \in \mathbb{N}$ the Nth *minimal worst case error* of $\mathrm{Int} = (\mathrm{Int}_d)_{d\in\mathbb{N}}$,

$$e(N, d; \mathrm{Int}_d, E_{d,\alpha}) := \inf_{\mathcal{A}_{N,d}} \sup_{\|f \,|\, E_{d,\alpha}\| \leq 1} \left|\mathrm{Int}_d(f) - \mathcal{A}_{N,d}(f)\right|,$$

equals the initial error $e(0, d; \mathrm{Int}_d, E_{d,\alpha}) = 1$ provided that $N < 2^d$. In other words, the integration problem on the full spaces $(E_{d,\alpha})_{d\in\mathbb{N}}$ suffers from the *curse of dimensionality*, since for every fixed $\varepsilon \in (0,1)$ its *information complexity* grows exponentially with the dimension d:

$$n(\varepsilon, d) := n(\varepsilon, d; \mathrm{Int}_d, E_{d,\alpha}) := \min\left\{N \in \mathbb{N}_0 \mid e(N, d; \mathrm{Int}_d, E_{d,\alpha}) \leq \varepsilon\right\} \geq 2^d, \quad d \in \mathbb{N}.$$

We generalize this result to the case of permutation-invariant[1] subspaces in the sense of [4]. To this end, for $d \in \mathbb{N}$ let $I_d \subseteq \{1, \ldots, d\}$ be some subset of coordinates and consider the integration problem $\mathrm{Int} = (\mathrm{Int}_d)_{d\in\mathbb{N}}$ restricted to the subspace $\mathfrak{S}_{I_d}(E_{d,\alpha})$ of all I_d*-permutation-invariant* functions $f \in E_{d,\alpha}$. That is, in dimension d we restrict ourselves to functions f that satisfy

$$f(\boldsymbol{x}) = f(\sigma(\boldsymbol{x})) \quad \text{for all } \boldsymbol{x} \in [0,1]^d \tag{2}$$

and any permutation σ from

$$\mathcal{S}_{I_d} := \left\{\sigma\colon \{1, \ldots, d\} \to \{1, \ldots, d\} \mid \sigma \text{ bijective and } \sigma|_{\{1,\ldots,d\}\setminus I_d} = \mathrm{id}\right\} \tag{3}$$

that leaves the elements in the complement of I_d fixed. For the ease of presentation we shall use the same notation for permutations $\sigma \in \mathcal{S}_{I_d}$ and for the corresponding permutations $\sigma'\colon \mathbb{R}^d \to \mathbb{R}^d$ of d-dimensional vectors, given by

$$\boldsymbol{x} = (x_1, \ldots, x_d) \mapsto \sigma'(\boldsymbol{x}) := (x_{\sigma(1)}, \ldots, x_{\sigma(d)}).$$

Observe that in the case $I_d = \emptyset$ we clearly have $\mathfrak{S}_{I_d}(E_{d,\alpha}) = E_{d,\alpha}$.

One motivation to study the integration problem restricted to those subspaces is related to approximate solutions of partial differential equations. Many approaches to obtain such solutions lead us to the problem of calculating high-dimensional integrals, e.g., to obtain certain wavelet coefficients. Obviously, it is of interest whether this can be done efficiently since taking into account a large number

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[1] In [4] we used the name *symmetric* what caused some confusion.

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