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On the Secant method

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ABSTRACT

We present a new semilocal convergence analysis for the Secant method in order to approximate a locally unique solution of a nonlinear equation in a Banach space setting. Our analysis is based on the weaker center-Lipschitz concept instead of the stronger Lipschitz condition which has been ubiquitously employed in other studies such as Amat et al. (2004) [2], Bosarge and Falb (1969) [9], Dennis (1971) [10], Ezquerro et al. (2010) [11], Hernández et al. (2005, 2000) [13,12], Kantorovich and Akilov (1982) [14], Laasonen (1969) [15], Ortega and Rheinboldt (1970) [16], Parida and Gupta (2007) [17], Potra (1982, 1984–1985, 1985) [18–20], Proinov (2009, 2010) [21,22], Schmidt (1978) [23], Wolfe (1978) [24] and Yamamoto (1987) [25] for computing the inverses of the linear operators. We also provide lower and upper bounds on the limit point of the majorizing sequences for the Secant method. Under the same computational cost, our error analysis is tighter than that proposed in earlier studies. Numerical examples illustrating the theoretical results are also given in this study.

1. Introduction

In this study, we are concerned with the problem of approximating a locally unique solution $x^\star$ of nonlinear equation

$$\mathcal{F}(x) = 0, \tag{1.1}$$

where $\mathcal{F}$ is a continuous operator defined on a nonempty convex subset $\mathbf{D}$ of a Banach space $\mathbf{X}$ with values in a Banach space $\mathbf{Y}$. Many problems from computational sciences and other disciplines can

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be expressed in the form of Eq. (1.1) using Mathematical Modeling [5,6,10,12,16]. The solution of these equations can rarely be found in closed form. That is why the most solution methods for these equations are iterative. The study about convergence analysis of iterative procedures is usually divided into two categories: semilocal and local convergence analysis. The semilocal convergence analysis derives convergence criteria from the information around an initial point while the local analysis finds estimates of the radii of convergence balls from the information around a solution.

The Secant method – also known under the names of Regular falsi or the method of chords – is one of the most used iterative methods for solving nonlinear equations. According to A.N. Ostrowski this method is known since the time of early Italian algebraists. It is well known that in the case of scalar equations, the Secant method is better than Newton's method from the point of view of efficiency index [7,6,14,16,18–23]. The Secant method was extended for the solution of Eq. (1.1) in Banach spaces by J.W. Schmidt [23]. They have used the notion of a divided difference of a nonlinear operator. In later studies, it was observed that it is possible to use the more general concept of consistent approximation of a derivative [7,6,16] instead of the notion of divided differences. In this case the Secant method can be described by the recurrence scheme

$$x_{n+1} = x_n - \delta\mathcal{F}(x_{n-1}, x_n)^{-1}\mathcal{F}(x_n), \quad \text{for each } n = 0, 1, 2, 3, \ldots \tag{1.2}$$

where $x_{-1}, x_0 \in \mathbf{D}$ are initial points and $\delta\mathcal{F}(x, y) \in \mathbb{L}(\mathbf{X}, \mathbf{Y})$ $(x, y \in \mathbf{D})$ is a consistent approximation of the Fréchet derivative of $\mathcal{F}$. $\mathbb{L}(\mathbf{X}, \mathbf{Y})$ denotes the space of bounded linear operators from $\mathbf{X}$ into $\mathbf{Y}$. The Secant method is an alternative to the well-known Newton's method

$$x_{n+1} = x_n - \mathcal{F}'(x_n)^{-1}\mathcal{F}(x_n), \quad \text{for each } n = 0, 1, 2, 3, \ldots$$

where $x_0 \in \mathbf{D}$ is an initial point. The following conditions are commonly associated with the semilocal convergence of the Secant method (1.2):

- $\mathcal{C}_1$: $\mathcal{F}$ is a nonlinear operator defined on a convex subset $\mathbf{D}$ of a Banach space $\mathbf{X}$ with values in a Banach space $\mathbf{Y}$,
- $\mathcal{C}_2$: x_{-1} and x_0 are two points belonging to the interior $\mathbf{D}^0$ of $\mathbf{D}$ and satisfying the inequality

$$\|x_0 - x_{-1}\| \le c,$$

- $\mathcal{C}_3$: $\mathcal{F}$ is Fréchet-differentiable on $\mathbf{D}^0$ and there exists a linear operator $\delta\mathcal{F} : \mathbf{D}^0 \times \mathbf{D}^0 \to \mathbb{L}(\mathbf{X}, \mathbf{Y})$ such that $\mathbf{A} = \delta F(x_{-1}, x_0)$ is invertible, its inverse $\mathbf{A}^{-1}$ is bounded,

$$\|\mathbf{A}^{-1}\,\mathcal{F}(x_0)\| \le \eta,$$
$$\|\mathbf{A}^{-1}\,(\delta\mathcal{F}(x, y) - \mathcal{F}'(z))\| \le \ell\,(\|x - z\| + \|y - z\|) \quad \text{for all } x, y, z \in \mathbf{D},$$
$$\overline{U}(x_0, r) = \{x \in \mathbf{X} : \|x - x_0\| \le r\} \subseteq \mathbf{D}^0,$$

for some $r > 0$ depending on ℓ, c, η and

$$\ell\, c + 2\sqrt{\ell\,\eta} \le 1. \tag{1.3}$$

Under condition (1.3), the R-order of convergence for the Secant method is $(1 + \sqrt{5})/2$. Several researchers, such as Amat et al. [2,1], Bosarge et al. [9], Dennis [10], Potra [18–20], Hernández et al. [9–11,13,12], Argyros [3–5,7,8,6], Parida et al. [17] and others [23,25,24], have used the conditions $(\mathcal{C}_1)$–$(\mathcal{C}_3)$ to provide sufficient convergence criteria for the Secant method. In many situations, the sufficient convergence criteria (1.3) may fail. For example, consider $\mathbf{X} = \mathbf{Y} = \mathbb{R}$, $x_0 = 1$, $\mathbf{D} = [p, 2-p]$ for $p \in (0, 1)$ and $c = 0.15$. Define function $\mathcal{F}$ on $\mathbf{D}$ by

$$\mathcal{F}(x) = x^3 - p.$$

For $p = 0.7252$ (see the computations in Section 4), we get

$$\eta = 0.102934888, \qquad l = 1.474985421$$

and

$$lc + 2\sqrt{l\eta} = 1.000548677 > 1.$$