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On the inverse of the discrepancy for infinite dimensional infinite sequences

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ABSTRACT

In 2001 Heinrich, Novak, Wasilkowski and Woźniakowski proved the upper bound $N^*(s,\varepsilon) \leq c_{\text{abs}} s\varepsilon^{-2}$ for the inverse of the star discrepancy $N^*(s,\varepsilon)$. This is equivalent to the fact that for any $N \geq 1$ and $s \geq 1$ there exists a set of N points in the s-dimensional unit cube whose star-discrepancy is bounded by $c_{\text{abs}}\sqrt{s}/\sqrt{N}$. Dick showed that there exists a double infinite matrix $(x_{n,i})_{n\geq 1, i\geq 1}$ of elements of $[0,1]$ such that for any N and s the star discrepancy of the s-dimensional N-element sequence $((x_{n,i})_{1\leq i\leq s})_{1\leq n\leq N}$ is bounded by

$$\frac{c_{\text{abs}}\sqrt{s\log N}}{\sqrt{N}}.$$

In the present paper we show that this upper bound can be reduced to $c_{\text{abs}}\sqrt{s}/\sqrt{N}$, which is (up to the value of the constant) the same upper bound as the one obtained by Heinrich et al. in the case of fixed N and s.

1. Introduction and statement of results

The star discrepancy $D_N^*(x_1,\ldots,x_N)$ of a sequence of points $(x_1,\ldots,x_N)$ from the s-dimensional unit cube is defined as

$$D_N^*(x_1,\ldots,x_N) = \sup_{I\subset[0,1]^s} \left| \lambda(I) - \frac{1}{N}\sum_{n=1}^{N} \mathbb{1}_I(x_n) \right|.$$

Here the supremum is taken over all axis-parallel boxes I which are contained in $[0,1]^s$ and have a vertex in the origin, and λ denotes the Lebesgue measure. The so-called *Quasi-Monte Carlo method* is

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based on the fact that point sequences having small discrepancy can be used for numerical integration. There exist many constructions of point sequences having small discrepancy, such as for example Halton sequences, Sobol sequences, etc. The discrepancy of the first N elements of such sequences (in dimension s) is bounded by $\mathcal{O}\left((\log N)^s N^{-1}\right)$, which is close to the optimal asymptotic order. However, discrepancy bounds of this type are only useful if the number of points N is very large in comparison with the dimension s. For this reason the notion of the *inverse of the discrepancy* was introduced: $N^*(s,\varepsilon)$ denotes the smallest possible number of points in the s-dimensional unit cube which have star discrepancy not exceeding ε. By a profound result of Heinrich et al. [13] we have

$$N^*(s,\varepsilon) \leq c_{\text{abs}} s \varepsilon^{-2},$$

which is equivalent to the fact that for any N and s there exists a sequence of N points in $[0,1]^s$ whose discrepancy is bounded by $c_{\text{abs}}\sqrt{s}/\sqrt{N}$ (c_{abs} denotes absolute constants, not always the same). Hinrichs [14] proved

$$N^*(s,\varepsilon) \geq c_{\text{abs}} s \varepsilon^{-1},$$

and thus the inverse of the star-discrepancy depends linearly on the dimension s. The dependence on ε is still an open problem.

The proof of Heinrich et al. uses a combinatorial result of Haussler, together with a result of Talagrand on empirical processes. In fact, what Heinrich et al. actually proved is the following: let $X_1,\dots,X_N$ be a sequence of independent, identically distributed (i.i.d.) $[0,1]^s$-uniformly-distributed random variables. Then with positive probability the discrepancy of $(X_1,\dots,X_N)$ is bounded by

$$c_{\text{abs}}\sqrt{s}/\sqrt{N}. \tag{1}$$

Extending this method, Dick [6] proved the existence of a (double infinite) matrix $(x_{n,i})_{n\geq 1, i\geq 1}$ of numbers $x_{n,i} \in [0,1]$ such that for any $N \geq 1$ and $s \geq 1$ the discrepancy of the s-dimensional N-element sequence $\left((x_{1,1},\dots,x_{1,s}),\dots,(x_{N,1},\dots,x_{N,s})\right)$ is bounded by

$$c_{\text{abs}}\sqrt{s\log N}/\sqrt{N}. \tag{2}$$

This means that there exist point sequences having small discrepancy, which can be extended both in the dimension s and the number of points N. This can be a significant advantage in applications. More precisely, Dick proved that a randomly generated double infinite matrix satisfies the aforementioned discrepancy bound with positive probability. This asymptotic upper bound contains an additional logarithmic factor in comparison with the estimate (1) for fixed N and s. However, it is clear that an entirely randomly generated matrix cannot achieve the bound (1) uniformly in N and s with positive probability, since by the Chung–Smirnov law of the iterated logarithm (see [20, p. 504]) already for the one-dimensional projections $(x_{1,1},\dots,x_{N,1})$ of such a matrix we have

$$\limsup_{N\to\infty} \frac{N D_N^*(x_{1,1},\dots,x_{N,1})}{\sqrt{2N\log\log N}} = \frac{1}{2} \quad \text{a.s.}$$

(the same asymptotic result holds for all s-dimensional projections for fixed s, see [19, Corollary 4.1.2]). Dick's result has been slightly improved by Doerr et al. [8], who obtained $c_{\text{abs}}\sqrt{s\log(1+N/s)}/\sqrt{N}$ instead of (2). In [3], we have further improved this upper bound to $\sqrt{c_{\text{abs}}s + c_{\text{abs}}\log\log N}/\sqrt{N}$, which is essentially the optimal upper bound which holds for a completely randomly generated matrix with positive probability.

The purpose of the present paper is to prove the existence of a double infinite matrix $(x_{n,i})_{n\geq 1, i\geq 1}$ such that the discrepancy of all its $N \times s$-dimensional projections is bounded by

$$c_{\text{abs}}\sqrt{s}/\sqrt{N}, \quad \text{for all } s \geq 1 \text{ and } N \geq 1.$$

This is the same upper bound as the one obtained by Heinrich et al. in the case of fixed N and s. Since such an upper bound cannot be achieved by an entirely randomly generated matrix, we will use a