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A primal–dual symmetric relaxation for homogeneous conic systems

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Abstract

We address the feasibility (existence of non-trivial solutions) of the pair of alternative conic systems of constraints

$$Ax = 0, \quad x \in C$$

and

$$-A^{\mathrm{T}}y \in C^*,$$

where $A \in \mathbb{R}^{m\times n}$, $m < n$, is a full row-rank matrix, and $C \subseteq \mathbb{R}^n$ is a closed convex cone. To this end, we reformulate the above pair of conic systems as a primal–dual pair of conic programs. Each of the conic programs corresponds to a natural relaxation of each of the two conic systems.

When C is a self-scaled cone with a known self-scaled barrier, the conic programming reformulation can be solved via an interior-point algorithm. For a well-posed instance A, a strict solution to one of the two original conic systems can be obtained in $O(\sqrt{v_C}\log(v_C\,\mathscr{C}(A))$ interior-point iterations. Here v_C is the complexity parameter of the self-scaled barrier of C and $\mathscr{C}(A)$ is Renegar's condition number of A. A central feature of our approach is the conditioning of the system of equations that arise at each interior-point iteration. The condition number of such system of equations grows in a controlled manner and remains bounded by a constant factor of $\mathscr{C}(A)^2$ throughout the entire algorithm.

Keywords: Conic programming; Condition numbers; Interior-point methods

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1. Introduction

We study the conic feasibility systems of constraints

$$Ax = 0, \quad x \in C \tag{1}$$

and

$$-A^{\mathrm{T}} y \in C^*, \tag{2}$$

where $A \in \mathbb{R}^{m\times n}$ is a full row-rank matrix, and $C \subseteq \mathbb{R}^n$ is a closed convex cone.

The conic systems (1), (2) are essentially *alternative* systems: either system is well-posed feasible if and only if the other does not have non-zero solutions.

We reformulate the above pair of feasibility problems as a primal–dual pair of conic programs. The conic programs correspond to natural relaxations of the original conic systems. When the cone C is self-scaled, the primal–dual reformulation can be solved via an interior-point algorithm. For a well-posed instance A, the algorithm determines which one of the conic systems is feasible and generates a strictly feasible solution.

A key feature of our approach is that the amount of computational work needed to solve the feasibility pair (1), (2) depends naturally on Renegar's condition number of the data instance A: the algorithm that we propose solves the above feasibility problem in a number of interior-point iterations that is proportional to $\mathcal{C}(A)$ (see Theorem 3.1). In addition, the condition number of the equations arising at each interior-point iteration is controlled throughout the algorithm and is bounded by a factor of $\mathcal{C}(A)^2$ (see Theorem 4.7). The first property, i.e., the complexity dependence on $\mathcal{C}(A)$ is in the same spirit as condition-based approaches to optimization and conic feasibility problems such as [5–7,15,17,18]. Indeed, our approach extends [17], where a purely "primal" approach for the feasibility problem $Ax = 0, x \in C$ was presented and analyzed in terms of $\mathcal{C}(A)$. The second property, i.e., the control of the condition numbers of the equations arising at each main iteration, provides a central foundational step for a rigorous study of the behavior of algorithms that perform finite precision arithmetic computations. Results of this type have been scarce in optimization, except for a few papers that address this issue in the linear programming case [3,4,22]. In all of these cases, a bound on the condition number of the equations arising at each main iteration is the crucial element at the heart of the roundoff analysis. The controlled conditioning of the equations arising in our approach paves the way for a formal study of the effects of finite-precision arithmetic for more general optimization problems. Indeed, in [2] we extend the scope of [3] to study the solvability of second-order cone feasibility problems under finite precision.

Some of our ideas are inspired by previous work by Peña and Renegar [17], and by Cucker and Peña [3]. In the former, a purely *primal* relaxation scheme for solving (1) was proposed and analyzed. In the latter, the authors devised a primal–dual scheme for solving (1) or (2) for the particular case when $C = \mathbb{R}^n_+$, using a finite-precision machine. This paper combines and extends both of these previous works. Unlike the purely primal approach used in [17], we reformulate the feasibility problem as a primal–dual pair of conic programs, in the same spirit as [3] but for more general conic systems. As a nice consequence of this primal–dual approach, *both* (1) *and* (2) are treated in a unified manner, without any a priori feasibility assumption of either system. The proofs presented herein are not only more general, but also more concise and transparent than the analogous ones in [3,17]. Our main results rely on a general perturbation theorem for self-scaled programs (Theorem 5.1) which is of independent interest. To ease our presentation, we develop the main technical ideas in the last section of the paper.