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Randomized approximation of Sobolev embeddings, III

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ABSTRACT

We continue the study of randomized approximation of embeddings between Sobolev spaces on the basis of function values. The source space is a Sobolev space with nonnegative smoothness order; the target space has negative smoothness order. The optimal order of approximation (in some cases only up to logarithmic factors) is determined. Extensions to Besov and Bessel potential spaces are given and a problem recently posed by Novak and Woźniakowski is partially solved. The results are applied to the complexity analysis of weak solution of elliptic PDE.

1. Introduction

In this paper we study the randomized approximation of Sobolev embeddings of $W_p^r(Q)$ into $W_q^s(Q)$, continuing the investigations from [10], where the case of $s = 0$ and Q being a cube was considered, and from [11], concerned with the case $s \geq 0$, Q a bounded Lipschitz domain. Now we deal with the case $s < 0$, again in general Lipschitz domains Q. We determine the optimal order of randomized approximation based on function values (sometimes only up to logarithmic factors). The results are new even for the case of Q being a cube and $p = q = 2$.

The case $s < 0$ is of interest in view of its role in weak solution of elliptic partial differential equations. We present some consequences in this direction.

The paper is organized as follows. In Section 3 we study the case $r = 0$. This is the essentially new situation, and we develop a multilevel Monte Carlo approximation algorithm. In Section 4 we combine it with the algorithm from [11] to cover the case of general r. The deterministic setting is discussed in Section 5, which also contains comparisons between the rates of deterministic and randomized approximation. In Section 6 we extend the results to other types of function spaces, which leads, in particular, to the solution of open problem 25 of Novak and Woźniakowski [15] for the case of standard information. Finally, in Section 7 an application to the complexity of weak solution of elliptic PDE is shown.

Many results are formulated in a slightly stronger way involving the dual of a Sobolev space with positive smoothness order as target space. These spaces are closely related to Sobolev spaces with

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negative smoothness order (see relation (127)), and the respective results for the latter are easily derived using duality (see Corollary 4.3 for Sobolev spaces and relations (171), (172), and Theorem 6.4 for the same situation in other function spaces).

2. Preliminaries

The paper is a direct continuation of [11]. Therefore we frequently use notation from there and refer the reader to [11] for explanation. For $1 \le p \le \infty$ we denote by p^* the dual exponent given by $1/p + 1/p^* = 1$. For a normed space X we denote the unit ball by $\mathcal{B}_X$ and the dual space by X^*. Throughout this paper log means $\log_2$.

We need some results on Banach space valued random variables. Given p with $1 \le p \le 2$, we recall from Ledoux and Talagrand [12] that the type p constant $\tau_p(Z)$ of a Banach space Z is the smallest c with $0 < c \le +\infty$, such that for all n and all sequences $(z_i)_{i=1}^n \subset Z$,

$$\mathbb{E}\left\|\sum_{i=1}^n \varepsilon_i z_i\right\|^p \le c^p \sum_{i=1}^n \|z_i\|^p, \tag{1}$$

where (ε_i) denotes a sequence of independent symmetric Bernoulli random variables on some probability space $(\Omega, \Sigma, \mathbb{P})$, i.e. $\mathbb{P}\{\varepsilon_i = 1\} = \mathbb{P}\{\varepsilon_i = -1\} = \frac{1}{2}$. Z is said to be of type p if $\tau_p(Z) < \infty$. Trivially, each Banach space is of type 1. Type p implies type p_1 for all $1 \le p_1 < p$. For $1 \le p < \infty$ all L_p spaces are of type $\min(p, 2)$. Moreover, the spaces ℓ_p^n are of type $\min(p, 2)$ uniformly in n, that is, $\tau_{\min(p,2)}(\ell_p^n) \le c$. Furthermore, $c_1(\log(n+1))^{1/2} \le \tau_2(\ell_\infty^n) \le c_2(\log(n+1))^{1/2}$.

We will use the following result. The case $p_1 = p$ of it is contained in Proposition 9.11 of [12]. The proof provided there easily extends to the case of general p_1 using some further tools from [12].

Lemma 2.1. *Let $1 \le p \le 2$, $p \le p_1 < \infty$. Then there is a constant $c > 0$ such that for each Banach space Z of type p, each $n \in \mathbb{N}$ and each sequence of independent, mean zero Z-valued random variables $(\zeta_i)_{i=1}^n$ with $\mathbb{E}\,\|\zeta_i\|^{p_1} < \infty$ $(1 \le i \le n)$ the following holds:*

$$\left(\mathbb{E}\left\|\sum_{i=1}^n \zeta_i\right\|^{p_1}\right)^{1/p_1} \le c\,\tau_p(Z)\left(\sum_{i=1}^n \left(\mathbb{E}\,\|\zeta_i\|^{p_1}\right)^{p/p_1}\right)^{1/p}.$$

Proof. Let $(\Omega, \Sigma, \mathbb{P})$ be the probability space that the ζ_i are defined on. Let $(\varepsilon_i)_{i=1}^n$ be independent, symmetric Bernoulli random variables on some probability space $(\Omega', \Sigma', \mathbb{P}')$ different from $(\Omega, \Sigma, \mathbb{P})$. We denote the expectation with respect to $\mathbb{P}'$ by $\mathbb{E}'$ (and the expectation with respect to $\mathbb{P}$, as before, by $\mathbb{E}$). Using first Lemma 6.3 of [12] and then the equivalence of moments (Theorem 4.7 of [12]), we get

$$\begin{aligned}\left(\mathbb{E}\left\|\sum_{i=1}^n \zeta_i\right\|^{p_1}\right)^{1/p_1} &\le 2\left(\mathbb{E}\,\mathbb{E}'\left\|\sum_{i=1}^n \varepsilon_i\zeta_i\right\|^{p_1}\right)^{1/p_1}\\ &\le 2c_{p,p_1}\left(\mathbb{E}\left(\mathbb{E}'\left\|\sum_{i=1}^n \varepsilon_i\zeta_i\right\|^{p}\right)^{p_1/p}\right)^{1/p_1},\end{aligned} \tag{2}$$

where the constant c_{p,p_1} depends only on p and p_1. Next we use the type inequality (1) and the triangle inequality in $L_{p_1/p}(\Omega, \mathbb{P})$ to obtain

$$\begin{aligned}\left(\mathbb{E}\left(\mathbb{E}'\left\|\sum_{i=1}^n \varepsilon_i\zeta_i\right\|^{p}\right)^{p_1/p}\right)^{1/p_1} &\le \tau_p(Z)\left(\mathbb{E}\left(\sum_{i=1}^n \|\zeta_i\|^p\right)^{p_1/p}\right)^{1/p_1}\\ &\le \tau_p(Z)\left(\sum_{i=1}^n \left(\mathbb{E}\,\|\zeta_i\|^{p_1}\right)^{p/p_1}\right)^{1/p}.\end{aligned} \tag{3}$$

Combining (2) and (3) completes the proof. □