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General local convergence theory for a class of iterative processes and its applications to Newton's method

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ABSTRACT

General local convergence theorems with order of convergence $r \geq 1$ are provided for iterative processes of the type $x_{n+1} = Tx_n$, where $T : D \subset X \to X$ is an iteration function in a metric space X. The new local convergence theory is applied to Newton iteration for simple zeros of nonlinear operators in Banach spaces as well as to Schröder iteration for multiple zeros of polynomials and analytic functions. The theory is also applied to establish a general theorem for the uniqueness ball of nonlinear equations in Banach spaces. The new results extend and improve some results of [K. Dočev, Über Newtonsche Iterationen, C. R. Acad. Bulg. Sci. 36 (1962) 695–701; J.F. Traub, H. Woźniakowski, Convergence and complexity of Newton iteration for operator equations, J. Assoc. Comput. Mach. 26 (1979) 250–258; S. Smale, Newton's method estimates from data at one point, in: R.E. Ewing, K.E. Gross, C.F. Martin (Eds.), The Merging of Disciplines: New Direction in Pure, Applied, and Computational Mathematics, Springer, New York, 1986, pp. 185–196; P. Tilli, Convergence conditions of some methods for the simultaneous computation of polynomial zeros, Calcolo 35 (1998) 3–15; X.H. Wang, Convergence of Newton's method and uniqueness of the solution of equations in Banach space, IMA J. Numer. Anal. 20 (2000) 123–134; I.K. Argyros, J.M. Gutiérrez, A unified approach for enlarging the radius of convergence for Newton's method and applications, Nonlinear Funct. Anal. Appl. 10 (2005) 555–563; M. Giusti, G. Lecerf, B. Salvy, J.-C. Yakoubsohn, Location and approximation of clusters of zeros of analytic functions, Found. Comput. Math. 5 (3) (2005) 257–311], and others.

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1. Introduction

In this paper we establish some general local convergence theorems with order of convergence $r \geq 1$ for iterative processes of the type

$$x_{n+1} = Tx_n, \quad n = 0, 1, 2, \ldots, \tag{1.1}$$

where $T: D \subset X \to X$ is an iteration function in a metric space X satisfying the following condition

$$E(Tx) \leq \varphi(E(x)) \quad \text{for all } x \in D \text{ with } Tx \in D \text{ and } E(x) \in J, \tag{1.2}$$

where $E: D \to \mathbb{R}_+$, J is an interval on $\mathbb{R}_+$ containing 0, φ is a gauge function on J, i.e. $\varphi: J \to J$. We prove that if T satisfies (1.2) and some other assumptions, then the initial convergence conditions of Picard iteration (1.1) can be given in the form $E(x_0) \in J$. This is way we call a function $E: D \to \mathbb{R}_+$ satisfying (1.2) *function of initial conditions* of T. Using this notion we establish a new local convergence theory for iterative processes of the type (1.1). Applying this theory one can get local convergence theorems for many iterations. In this work we apply our theory to Newton iteration as well as to Schröder iteration (Newton iteration for multiple zeros) with respect to various functions of initial conditions.

The paper is structured as follows. The general local convergence theory for the iterative processes of the type (1.1) is presented in Sections 2 and 3. The main results here are formulated in Theorems 3.6 and 3.8. In Section 4 we apply Theorem 3.6 to Newton iteration for multiple zeros of a complex polynomial $f(z)$ with respect to the function of initial conditions E defined as follows $E(z) = |z-\xi|/d$, where ξ is a zero of f (simple or multiple) and d denotes the distance from ξ to the other zeros of f. The results in this section extend the corresponding results of Dočev [5]. In Section 5 we continue to study the local convergence of Newton iteration for multiple polynomial zeros but with respect to the function of initial conditions defined by $E(z) = |z - \xi|/\rho(z)$, where $\rho(z)$ denotes the distance from z to the nearest zero of f which is not equal to ξ. The results of this section improve and extend a result of Tilli [18]. In Section 6 we apply our theory to Newton iteration for multiple zeros of a complex function $f(z)$ which is analytic in a neighborhood of ξ, where ξ is a zero of f with multiplicity $m \in \mathbb{N}$. The function of initial conditions here is defined as follows $E(z) = \gamma(\xi)\|z - \xi\|$, where $\gamma(\xi) = \sup_{k>m} \left|\frac{m! f^{(k)}(\xi)}{k! f^{(m)}(\xi)}\right|^{1/(k-m)}$. This quantity $\gamma(\xi)$ has been introduced in the case $m = 1$ by Smale [17] and in the case $m \geq 1$ by Yakoubsohn [24]. The first result for the convergence ball of Newton iteration with respect to this function of initial conditions is due to Traub and Woźniakowski [19] for simple zeros of analytic functions (even in Banach spaces). Later, Smale [17] (γ-theorem) improves Traub and Woźniakowski's result. In 2005, Giusti, Lecerf, Salvy and Yakoubsohn [6, Proposition 3.4] generalize γ-theorem to cluster of zeros. In this section we improve Proposition 3.4 of [6] in the case of multiple zeros of analytic functions. In Section 7 we investigate the local convergence of Newton iteration $x_{n+1} = x_n - F'(x_n)^{-1}F(x_n)$ for a simple zero ξ of a nonlinear Fréchet differentiable operator F defined on a subset D of a Banach space X with values in a Banach space Y. Here we study Newton iteration with respect to the standard function of initial conditions $E(x) = \|x - \xi\|$. The main result in this section gives a unified theory for the convergence ball of Newton's method and extends the corresponding results of Traub and Woźniakowski [19], Smale [17], Wang [21], Wang and Li [23],