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A note on a modification of Moser's method

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Abstract

We use a recurrence technique to obtain semilocal convergence results for Ulm's iterative method to approximate a solution of a nonlinear equation $F(x) = 0$

$$\begin{cases} x_{n+1} = x_n - B_n F(x_n), & n \geqslant 0, \\ B_{n+1} = 2B_n - B_n F'(x_{n+1}) B_n, & n \geqslant 0. \end{cases}$$

This method does not contain inverse operators in its expression and we prove it converges with the Newton rate. We also use this method to approximate a solution of integral equations of Fredholm-type.

MSC: 45G10; 47H17; 65J15

Keywords: Iterative processes; *R*-order of convergence; Semilocal convergence

1. Introduction

In this paper we consider an operator F defined in an open, convex and nonempty subset Ω of a Banach space X with values in another Banach space Y.

We consider the problem of approximating a solution x^* of a nonlinear equation

$$F(x) = 0. \tag{1}$$

Without any doubt Newton's method is the most used iterative process to solve this problem. It is given by the algorithm: $x_{n+1} = x_n - F'(x_n)^{-1} F(x_n)$, $n \geqslant 0$ for x_0 given. This iterative process

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has quadratic R-order of convergence so its speed of convergence and its operational cost is quite balanced.

Other methods, such as higher order methods also include in their expression the inverse of the operator F'. To avoid this problem, Newton-type methods: $x_{n+1} = x_n - H_n F(x_n)$, where H_n is an approximation of $F'(x_n)^{-1}$ are considered. One of these methods was proposed by Moser in [4] in this way. Given $x_0 \in \Omega$ and $A_0 \in \mathcal{L}(Y, X)$, the following sequences are defined

$$x_{n+1} = x_n - A_n F(x_n), \quad n \geqslant 0, \tag{2}$$

$$A_{n+1} = A_n - A_n(F'(x_n)A_n - I_X), \quad n \geqslant 0, \tag{3}$$

where I_X is the identity operator in X. The first equation is similar to Newton's method, but replacing the operator $F'(x_n)^{-1}$ by a linear operator A_n. The second equation is Newton's method applied to equation $g_n(A) = 0$ where $g_n : \mathcal{L}(Y, X) \to \mathcal{L}(X, Y)$ is defined by $g_n(A) = A^{-1} - F'(x_n)$. So $\{A_n\}$ gives us an approximation of $F'(x_n)^{-1}$.

In addition, it can be shown that the rate of convergence for the above scheme is $(1 + \sqrt{5})/2$, provided the root of (1) is simple [4]. However, this is unsatisfactory from a numerical point of view because the scheme uses the same amount of information per step as Newton's method, yet, it converges no faster than the secant method.

Moser's method was developed as a technical tool for investigating the stability of the N-body problem in celestial mechanics. The main difficulty in this, and similar problems involving small divisors, is the solution of a system of nonlinear partial differential equations. In essence, Moser's idea is to solve the problem by a sequence of changes of variables.

In [10] Ulm proposed the following iterative method to solve nonlinear equations. Given $x_0 \in \Omega$ where F is a Fréchet-differentiable operator and $B_0 \in \mathcal{L}(Y, X)$, Ulm defines

$$\begin{cases} x_{n+1} = x_n - B_n F(x_n), & n \geqslant 0, \\ B_{n+1} = 2B_n - B_n F'(x_{n+1}) B_n, & n \geqslant 0. \end{cases} \tag{4}$$

Notice that, here $F'(x_{n+1})$ appears instead of $F'(x_n)$ in (3). This is crucial for obtaining fast convergence. Under certain assumptions, Ulm showed, that the method generates successive approximations that converge to a solution of (1), asymptotically as fast as Newton's method. Ulm applied the method to several particular classes of operator equations.

The method exhibits several attractive features. First, it converges with the Newton rate. Second, it is inverse free: you do not need to solve a linear equation at each iteration. Third, in addition to solve the nonlinear equation (1), the method produces successive approximations $\{B_n\}$ to the value of $F'(x^*)^{-1}$, being x^* a solution of (1). This property is very helpful when one investigates the sensitivity of the solution to small perturbations.

Although method (4) was firstly proposed by Ulm [10], it has been also considered by other authors. For instance, Hald [1] showed the quadratic convergence of the method. Later, Zehnder [13] or Petzeltova [5] have studied the convergence of the method under Kantorovich-type conditions.

An alternative to Kantorovich theory to study the convergence of iterative processes to solve nonlinear equations is given by the known as Smale's point estimate theory [8,9]. Roughly speaking, if x_0 is an initial value such that the sequence $\{x_n\}$ satisfies

$$\|x_n - x^*\| \leqslant \left(\frac{1}{2}\right)^{2^n-1} \|x_0 - x^*\|$$