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Slow motion for the 1D Swift–Hohenberg equation

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Abstract

The goal of this paper is to study the behavior of certain solutions to the Swift–Hohenberg equation on a one-dimensional torus $\mathbb{T}$. Combining results from Γ-convergence and ODE theory, it is shown that solutions corresponding to initial data that is L^1-close to a jump function v, remain close to v for large time. This can be achieved by regarding the equation as the L^2-gradient flow of a second order energy functional, and obtaining asymptotic lower bounds on this energy in terms of the number of jumps of v.
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1. Introduction, motivation and main results

The fourth order partial differential equation

$$u_t = ru - (\bar{q}^2 + \partial_x^2)^2 u + f(u) \tag{1.1}$$

is a generalization of the Swift–Hohenberg equation introduced in 1977 by Swift and Hohenberg [36] as a model for the study of pattern formation, in connection with the Rayleigh–Bérnard convection, e.g. see [13,27]. Among many different applications, the most famous ones in the literature are those in connection to pattern formation in vibrated granular materials [37], buckling of long elastic structures [24], Taylor–Couette flow [23,32], and in the study of lasers [28]. Moreover, in recent years great attention has been paid to models of phase transitions in the study

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of pattern-formation in bilayer membranes, see e.g. [11] where the Swift–Hohenberg equation turns out to be the gradient flow of Ginzburg–Landau type energies, with respect to the right inner product structure.

Consider (1.1) on a periodic domain with a characteristic size $L = 1/\varepsilon$, where $0 < \varepsilon \ll 1$. Letting W be the primitive of $s \mapsto 2(f(s) + (r - \bar{q}^4)s)$, $q := 2\bar{q}^2$, and rescaling time and space by ε in (1.1) one arrives at the rescaled form

$$\begin{cases} u_t = -W'(u) - 2\varepsilon^2 q u_{xx} - 2\varepsilon^4 u_{xxxx} & x \in \mathbb{T}, t > 0, \\ u(x,0) = u_{0,\varepsilon}(x) & x \in \mathbb{T}, \end{cases} \tag{1.2}$$

where $\mathbb{T}$ is a one-dimensional torus. We assume that $W : \mathbb{R} \to [0, +\infty)$ is a double-well potential with phases supported at -1 and 1, and we study the long-time behavior of solutions when $q > 0$ is sufficiently small. In particular, due to the presence of the small parameter ε in (1.2) the solutions are expected to develop interfacial structure driven by the minima of the potential W. Equation (1.2) may be viewed as a gradient flow associated to a second order energy functional, and our main result consists of an asymptotic lower bound on the corresponding energy functional and the consequent bounds on the speed of evolution of the developed interfaces. Below we outline interfacial dynamics results for the lower order Allen–Cahn equation and its generalizations that provide much of the motivation for our analysis.

1.1. Allen–Cahn equation and generalizations to higher order

Equations displaying interfacial dynamics have been studied extensively in the last two decades. The prototypical example is the Allen–Cahn equation

$$u_t = \varepsilon^2 u_{xx} - W'(u), \quad x \in I, \ t > 0, \tag{1.3}$$

(as well as its higher dimensional analog) seen as the L^2-gradient flow of the energy

$$G_\varepsilon(u; I) := \int_I \left(\frac{1}{\varepsilon} W(u) + \frac{\varepsilon}{2} |u_x|^2 \right) dx, \quad u \in H^1(I), \tag{1.4}$$

where $I \subset \mathbb{R}$ is an interval. The special gradient-flow structure of (1.3) has allowed its analysis by a wide variety of methods and techniques.

In particular, it has been shown for the Allen–Cahn equation (see [9] and the references therein) that if $\varepsilon \ll 1$ the evolution from a sufficiently regular initial data occurs in four main stages. In the first stage, the diffusion term $\varepsilon^2 u_{xx}$ can be ignored and the leading order dynamics are driven by the ε independent ordinary differential equation $u_t = -W'(u)$. This is the time-scale in which interfaces develop, i.e., regions in the space domain that separate almost constant solutions corresponding to the stable equilibria of the ordinary differential equation. This stage, referred to as the generation of interface, has been analyzed for the Allen–Cahn equation first in [16], and subsequently in [9,10,14,35], and other papers.

As the regions separating unequal equilibria decrease in length, the spacial gradient necessarily increases, and after $O(|\ln \varepsilon|)$ time the dynamics are driven by a balance between the two terms on the right-hand side of (1.3). In particular, as shown in [9], after $O(\varepsilon^{-1})$ time the solution is exponentially close to the standing-wave profile

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