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J. Differential Equations 262 (2017) 116–144

Journal of Differential Equations

www.elsevier.com/locate/jde

Short-range scattering of Hartree type fractional NLS

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Received 1 February 2016; revised 18 July 2016

Available online 23 September 2016

Abstract

In this paper we consider scattering problem for Hartree type fractional NLS with $|\nabla|^\alpha$ $(1 < \alpha < 2)$ and potential $V \sim |x|^{-\gamma}$. We show small data scattering in a weighted space for the short range $\frac{6-2\alpha}{4-\alpha} < \gamma < 2$. The difficulty arises from the non-locality and non-smoothness of $|\nabla|^\alpha$. To overcome it we utilize the method of commutator estimate based on Balakrishnan's formula.

MSC: 35Q55; 35Q40

Keywords: Fractional NLS; Hartree type potential; Short-range interaction; Small data scattering

1. Introduction

In this paper we consider a small data scattering problem of the fractional nonlinear Schrödinger equation (fNLS) with Hartree type potential:

$$\begin{cases} i\partial_t u = |\nabla|^\alpha u + (V * |u|^2)u \text{ in } \mathbb{R}^{1+d}, \\ u(0) = \varphi \in H^s(\mathbb{R}^d), \end{cases} \tag{1.1}$$

where $d \ge 1$, $1 < \alpha < 2$, $s \ge 0$ and V is a complex-valued measurable function on $\mathbb{R}^d$. Here $|\nabla|^\alpha = (-\Delta)^{\frac{\alpha}{2}} = \mathcal{F}^{-1}|\xi|^\alpha\mathcal{F}$ is the fractional derivative of order α and $*$ denotes the space convolution. By Duhamel's formula, (1.1) is written as an integral equation

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http://dx.doi.org/10.1016/j.jde.2016.09.025

$$u(t) = e^{-it|\nabla|^\alpha}(\varphi) - i\int_0^t e^{-i(t-t')|\nabla|^\alpha}(V * |u(t')|^2)u(t')\,dt', \tag{1.2}$$

where the linear propagator $e^{-it|\nabla|^\alpha} f$ is the solution to the linear problem $i\partial_t z = |\nabla|^\alpha z$ with initial datum f. Then it is formally given by

$$e^{-it|\nabla|^\alpha} f = \mathcal{F}^{-1} e^{-it|\xi|^\alpha} \mathcal{F} f = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i(x\cdot\xi - t|\xi|^\alpha)} \widehat{f}(\xi)\,d\xi. \tag{1.3}$$

Here $\widehat{f} = \mathcal{F} f$ is the Fourier transform of f such that $\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\,dx$ and we denote its inverse Fourier transform by $\mathcal{F}^{-1} g(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{ix\cdot\xi} g(\xi)\,d\xi$.

During the decade fractional NLS including semi-relativistic equations have been extensively studied to describe natural phenomena in the context of fractional quantum mechanics [18,19], system of bosons [11], system of long-range lattice interaction [16], water waves [22], turbulence [3] and so on. Heuristically, Hartree nonlinearity can be interpreted as an interaction between particles with potential V [11]. The potential V taken into account is of short-range interaction as follows:

(S-R1) V is differentiable on $\mathbb{R}^d \setminus \{0\}$ and for $1 < \gamma < d$ it satisfies that

$$|V(x)| + |x||\nabla V(x)| \le C|x|^{-\gamma}, \quad \forall x \ne 0. \tag{1.4}$$

If $0 < \gamma \le 1$, then V is referred to be of long-range interaction. We adopt the terminologies short-/long-range interaction from the scattering theory of Hartree equations ($\alpha = 2$).

In this paper we are concerned with small data scattering of (1.1) for a class of potentials V.

Definition 1.1. We say a solution u to (1.1) scatters (to $u_\pm$) in a Hilbert space $\mathcal{H}$ if there exist $\varphi_\pm \in \mathcal{H}$ (with $u_\pm(t) = e^{-it|\nabla|^\alpha}\varphi_\pm$) such that

$$\lim_{t\to\pm\infty} \|e^{it|\nabla|^\alpha} u(t) - \varphi_\pm\|_{\mathcal{H}} = 0.$$

If V has a long range, it was shown in [6] that scattering may not occur even in L^2. The short-range scattering in H^s can be shown simply by Strichartz estimates on the real line when $2 < \gamma < d$ and $s > s_* := \frac{\gamma-\alpha}{2}$ since the dispersion of solution is fast. This is also the case for Hartree and semi-relativistic equation. See [12,8,13,5]. But in case when $1 < \gamma \le 2$, the dispersion of solution to (1.1) is not good for Strichartz estimate on the whole time. One may usually think of two ways (radial symmetry, weighted normed space) to observe scattering. Under the radial assumption the global Strichartz estimate can cover the range $1 < \gamma \le 2$ in part. To be more precise, small data scattering in H^{s_*} is possible when α, γ are restricted to $\frac{2d}{2d-1} \le \alpha < 2$ and $\alpha \le \gamma < d$. For this see [5]. In [7] even a large data scattering in energy space is treated under radial symmetry when $\gamma = 2\alpha$ (energy-critical) and $\frac{2d}{2d-1} < \alpha < 2$. The other way is to use a weighted normed space as in Hartree [14] and semi-relativistic [13] equations. If the initial data is in a weighted space, then the solution could be dispersive enough to scatter.