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Asymptotic behavior and decay estimates of the solutions for a nonlinear parabolic equation with exponential nonlinearity

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Abstract

We consider a nonlinear parabolic equation with an exponential nonlinearity which is critical with respect to the growth of the nonlinearity and the regularity of the initial data. After showing the equivalence of the notions of weak and mild solutions, we derive decay estimates and the asymptotic behavior for small global-in-time solutions.

Keywords: Asymptotic behavior; Nonlinear heat equation; Fractional diffusion equation; Exponential nonlinearity; Orlicz space

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1. Introduction

We consider the Cauchy problem for the semilinear heat equation

$$\begin{cases} \partial_t u = \Delta u + f(u), & t > 0,\ x \in \mathbb{R}^n, \\ u(0,x) = \varphi(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $n \geq 1$, $u(t,x) : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ is the unknown function, $\partial_t = \partial/\partial t$, $f(u)$ contains the nonlinearity and φ is the initial data. This paper is concerned with the asymptotic behavior and decay estimates of the solutions of (1.1) in certain limiting cases which are critical with respect to the growth of the nonlinearity and the regularity of the initial data.

Before introducing the subject of this paper, let us recall some related known results.

The polynomial case. The case of power nonlinearities $f(u) = |u|^{p-1}u$ with $p > 1$, that is

$$\begin{cases} \partial_t u = \Delta u + |u|^{p-1}u, & t > 0,\ x \in \mathbb{R}^n, \\ u(0,x) = \varphi(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.2}$$

has been extensively studied since the pioneering works by Fujita [6] and Weissler [22,23]. It is well-known that the problem (1.2) satisfies a scale invariance property. In fact, for $\lambda \in \mathbb{R}_+$, if u is a solution of (1.2), then

$$u_\lambda(t,x) := \lambda^{\frac{2}{p-1}} u(\lambda^2 t, \lambda x) \tag{1.3}$$

is also a solution of (1.2) with initial data $\varphi_\lambda(x) := \lambda^{2/(p-1)}\varphi(\lambda x)$. So, all function spaces invariant with respect to the scaling transformation (1.3) play a fundamental role in the study of the Cauchy problem (1.2). In the framework of Lebesgue spaces, we can easily show that the norm of the space $L^q(\mathbb{R}^n)$ is invariant with respect to (1.3) if and only if $q = q_c$ with $q_c = n(p-1)/2$, and it is well-known that, given the nonlinearity $|u|^{p-1}u$, the Lebesgue space $L^{q_c}(\mathbb{R}^n)$ plays the role of *critical space* for the well-posedness of (1.2) (see e.g. [2,7,22,23]).

Indeed, for any $q \geq q_c$ and $q > 1$, or $q > q_c$ and $q \geq 1$ (*subcritical case*), and for any initial data $\varphi \in L^q(\mathbb{R}^n)$ there exists a local (in time) solution of the Cauchy problem (1.2). On the other hand, for initial data belonging to $L^q(\mathbb{R}^n)$ with $1 < q < q_c$ (*supercritical case*), Weissler [22] and Brezis–Cazenave [2] indicate that for certain $\varphi \in L^q(\mathbb{R}^n)$ there exists no local (in time) solution in any reasonable sense.

We can also state the previous results from a different point of view. Given any initial data in the Lebesgue space L^q, $1 < q < +\infty$, the Cauchy problem (1.2) is well-posed if and only if the power nonlinearity p is smaller than or equal to the critical value $p_c = 1 + (2q)/n$. Moreover, the *critical case*, which is defined equivalently by $q = q_c$ or $p = p_c$, is the only case for which global (in time) existence can be established for small initial data.

The same polynomial nonlinearity has also been considered for the Schrödinger equation

$$\begin{cases} i\partial_t u + \Delta u = |u|^{p-1}u, & t > 0,\ x \in \mathbb{R}^n, \\ u(0,x) = \varphi(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.4}$$

Here $u : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{C}$ and $\varphi \in H^s(\mathbb{R}^n)$ with $0 \leq s < n/2$. Also for problem (1.4) the scaling invariance (1.3) holds, and the scaling argument indicates the critical Sobolev space and exponent