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Blow-up rates of solutions of initial-boundary value problems for a quasi-linear parabolic equation

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Abstract

We consider initial-boundary value problems for a quasi linear parabolic equation, $k_t = k^2(k_{\theta\theta} + k)$, with zero Dirichlet boundary conditions and positive initial data. It has known that each of solutions blows up at a finite time with the rate faster than $\sqrt{(T-t)^{-1}}$. In this paper, it is proved that $\sup_\theta k(\theta, t) \approx \sqrt{(T-t)^{-1} \log\log(T-t)^{-1}}$ as $t \nearrow T$ under some assumptions. Our strategy is based on analysis for curve shortening flows that with self-crossing brought by S.B. Angenent and J.J.L. Velázquez. In addition, we prove some of numerical conjectures by Watterson which are keys to provide the blow-up rate.

Keywords: Type II blow-up; Quasi-linear parabolic equations; Curve shortening flows

1. Introduction

Consider solutions of a partial differential equation

$$k_t(\theta, t) = k(\theta, t)^2 \Big(k_{\theta\theta}(\theta, t) + k(\theta, t) \Big) \quad \text{for } \theta \in (-L, L) \text{ and } 0 < t < T \tag{1.1}$$

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with initial-boundary value conditions

$$\begin{cases} k(\pm L, t) = 0 & \text{for } 0 \le t < T, \\ k(\theta, 0) = k_0(\theta) & \text{for } \theta \in (-L, L), \end{cases} \tag{1.2}$$

where $L > \dfrac{\pi}{2}$, $T > 0$ and k_0 is a smooth function in $(-L, L)$ and continuous in $[-L, L]$ with $k_0(\pm L) = 0$. This problem arises in a model for the resistive diffusion of a force-free magnetic field in a plasma confined between two walls (for details, see [13,14], and so forth).

It is well-known results in appropriate initial data that if $L > \dfrac{\pi}{2}$ then there exists $T > 0$ such that each of solutions to (1.1)–(1.2) satisfies

$$\limsup_{t \nearrow T} \sup_{\theta \in (-L,L)} (T - t)^{\frac{1}{2}} k(\theta, t) = \infty. \tag{1.3}$$

(See [3,9,18].) This implies that all of solutions of (1.1)–(1.2) have blow-up rates of Type II, that is, they blow up faster than any self similar solutions of (1.1). Furthermore, the numerical conjecture have provided as follows:

$$\sup_{\theta \in (-L,L)} k(\theta, t) \approx \left(\frac{1}{T - t} \log\log \frac{1}{T - t} \right)^{\frac{1}{2}} \quad \text{as } t \nearrow T \tag{1.4}$$

(see [9,11,17]), where $T > 0$ is the blow-up time of a solution given in (1.3). Our purpose of this paper is investigate the blow-up rates for solutions of (1.1)–(1.2). As related works for our purpose, we refer some important results for curvatures of shrinking convex curves in curvature flows. Precisely, consider a space of parameters $P \subset \mathbb{R}$ and a family of immersed curves $\mathcal{X}$: $P \times [0, T) \to \mathbb{R}^2$ with positive curvatures shrinking under conditions of

$$\frac{\partial \mathcal{X}}{\partial t} = -k\mathcal{N} \quad \text{in } P \times (0, T), \tag{1.5}$$

where k and $\mathcal{N}$ is the curvature and outer unit normal of the immersed curve $C_t := \{\mathcal{X}(u, t) \,|\, u \in P, t \in [0, T)\}$, respectively.

When $\mathcal{N}(u, t) = (\cos\theta(u, t), \sin\theta(u, t))$ at a point $\mathcal{X}(u, t)$, that is, $\theta = \theta(u, t)$ is the angle of the outer normal at $\mathcal{X}(u, t)$, it is easily verified that if $k > 0$ on $P \times [0, T)$ then k can be parametrized by θ and $k = k(\theta, t)$ is periodic and satisfies (1.1) with $L = \infty$. When $P = \mathbb{S}^1$ and $\{C_t\}$ is a family of strictly convex closed curves, it was proved in [10] that curvatures blow up at a finite time ($T > 0$) and there exists a constant $C > 0$ such that $\lim_{t \nearrow T} (T - t)^{\frac{1}{2}} k(\theta, t) = C$.

[4] and [6] deal with the case of $P = (\mathbb{R}/2\nu\pi)\mathbb{Z}$ ($\nu \in \mathbb{N}$) that C_t has a self-crossing point (the left in Fig. 1). It was shown in [4] that C_t tends to a cardioid-like curve which has a singular point. Here, "singular" means that the curvature at this point blows up to $+\infty$ (the right in Fig. 1). Furthermore, [6] provided a result of the blow-up rate for this case in special initial conditions that the maximum of curvatures at singular points blows up with the rate equal to (1.4).

In the term of curvature flows, we can regard solutions of (1.1)–(1.2) as curvatures of curves evolving from strophoid-like based on (1.5) (Fig. 2). Remark that curvatures at any points on curves in Fig. 2 are positive and closed to 0 as the right and left terminal edges extend to $\pm\infty$.