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Rough differential equations with unbounded drift term

S. Riedel *, M. Scheutzow

Institut für Mathematik, Technische Universität Berlin, Germany

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Abstract

We study controlled differential equations driven by a rough path (in the sense of T. Lyons) with an additional, possibly unbounded drift term. We show that the equation induces a solution flow if the drift grows at most linearly. Furthermore, we show that the semiflow exists assuming only appropriate one-sided growth conditions. We provide bounds for both the flow and the semiflow. Applied to stochastic analysis, our results imply *strong completeness* and the existence of a stochastic (semi)flow for a large class of stochastic differential equations. If the driving process is Gaussian, we can further deduce (essentially) sharp tail estimates for the (semi)flow and a Freidlin–Wentzell-type large deviation result.

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0. Introduction

T. Lyons' theory of *rough paths* can be used to solve controlled ordinary differential equations (ODE) of the form

* Corresponding author.
E-mail addresses: riedel@math.tu-berlin.de (S. Riedel), ms@math.tu-berlin.de (M. Scheutzow).

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$$\begin{aligned} dy &= b(y)\,dt + \sum_{i=1}^{d} \sigma_i(y)\,dx_t^i; \quad t \in [0,T] \\ y_0 &= \xi \in \mathbb{R}^m \end{aligned} \tag{0.1}$$

for vector fields $b, \sigma_1, \dots, \sigma_d \colon \mathbb{R}^m \to \mathbb{R}^m$ and non-differentiable, $1/p$-Hölder continuous paths $x \colon [0,T] \to \mathbb{R}^d$. However, one of Lyons' key insights was that the equation (0.1) as it stands is ill-posed[1] in the case of $p \geq 2$. Instead, one has to enhance the path $x \colon [0,T] \to \mathbb{R}^d$ with additional information (which can be interpreted as its iterated integrals) to a path $\mathbf{x}$ taking values in a larger space. Defining a suitable (p-variation or Hölder-type) topology on this space of paths allows to solve the corresponding "lifted" equation

$$\begin{aligned} dy &= b(y)\,dt + \sigma(y)\,d\mathbf{x}_t; \quad t \in [0,T] \\ y_0 &= \xi \in \mathbb{R}^m \end{aligned} \tag{0.2}$$

uniquely in the way that the solution map (also called *Itō–Lyons map*) $\mathbf{x} \mapsto y$ is continuous. This paves way to a genuine *pathwise* stochastic calculus for a huge class of (not-necessarily martingale-type) driving signals (cf. e.g. [8, Chapters 13–20] and the references therein). Rough paths theory is now well-established, and since Lyons' seminal article [16], several monographs have appeared (cf. [14,11,8,6]) which expose the theory and its various applications. Let us also briefly mention that rough paths ideas were used by M. Hairer to solve stochastic partial differential equations (SPDE) like the KPZ-equation (see [9]) and form an important part in his theory of *regularity structures* (cf. [10] and [6] where the link between rough paths and regularity structures is explained).

In the present work, we aim to solve (0.2) for a general, possibly unbounded drift term b while we assume σ to be bounded and sufficiently smooth. In the literature about rough paths, a convenient way to take care of the drift part is to regard $t \mapsto t$ as an additional (smooth) component of the rough path $\mathbf{x}$, and b as another component of σ (cf. e.g. [6, Exercise 8.15]). However, this yields unnecessary smoothness assumptions, and allowing b to be unbounded leads to the study of general unbounded vector fields for rough differential equations (which is a delicate topic, cf. [12] for a discussion). Maybe more important, the bounds for the solution y which are available in this case (cf. e.g. [8, Exercise 10.56]) are bounds which grow exponentially in the rough path norm of $\mathbf{x}$, whereas bounded diffusion vector fields should yield polynomial bounds. The main theorems in the present paper (Theorem 3.1 and Theorem 4.3) provide exactly the bounds expected.

A rough differential equation can be seen as a special case of a non-autonomous ordinary differential equation. Therefore, it should not come as a big surprise that such equations naturally induce continuous two parameter flows[2] on the state space $\mathbb{R}^m$ (at least if all vector fields are bounded, cf. [13], [8, Section 11.2], [6, Section 8.9]). Note that this immediately implies that a stochastic differential equation (SDE) induces a stochastic flow provided the driving process has sample paths in a rough paths space (which is the case, for instance, for a Brownian motion). In particular, the SDE is *strongly complete* which means that it can be solved globally on a set

[1] More precisely, Lyons showed that the map assigning to each smooth path x the solution y to the ordinary differential equation (0.1) is not closable in the space of p-variation or $1/p$-Hölder continuous paths.

[2] In fact, in [1], the flow is even the central object of interest and it is constructed directly, skipping the intermediate step of defining the solution to (0.2) for a fixed initial datum ξ first.