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On unified theory for scalar conservation laws with fluxes and sources discontinuous with respect to the unknown ☆

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Abstract

We deal with the Cauchy problem for multi-dimensional scalar conservation laws, where the fluxes and the source terms can be discontinuous functions of the unknown. The main novelty of the paper is the introduction of a kinetic formulation for the considered problem. To handle the discontinuities we work in the framework of re-parametrization of the flux and the source functions, which was previously used for Kružkov entropy solutions. Within this approach we obtain a fairly complete picture: existence of entropy measure valued solutions, entropy weak solutions and their equivalence to the kinetic solution. The results of existence and uniqueness follow under the assumption of Hölder continuity at zero of the flux. The source term, what is another novelty for the studies on problems with discontinuous flux, is only assumed to be one-side Lipschitz, not necessarily monotone function.

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1. Introduction

We focus on the Cauchy problem for a scalar hyperbolic balance law of the following form

$$\begin{aligned} \partial_t u + \operatorname{div} \mathbf{F}(u) &= G(u) \quad \text{in } \mathbb{R}^{d+1}_+, \\ u(0) &= u_0 \qquad \text{in } \mathbb{R}^d, \end{aligned} \tag{1.1}$$

where $\mathbb{R}^{d+1}_+ := (0, \infty) \times \mathbb{R}^d$, $d \geq 1$ denotes an arbitrary spatial dimension, $u : \mathbb{R}^{d+1}_+ \to \mathbb{R}$ is an unknown function, $\mathbf{F} : \mathbb{R} \to \mathbb{R}^d$ is a given flux of the quantity u, $u_0 : \mathbb{R}^d \to \mathbb{R}$ is the initial condition and $G : \mathbb{R} \to \mathbb{R}$ is the given source term. In addition we assume that u vanishes as $|x| \to \infty$. The main goal of the paper is to build a sufficiently robust framework that is capable to cover as general class of fluxes and source terms as possible. In particular, we want to focus on the cases when both quantities are discontinuous functions of the unknown u. The starting point is the method introduced in [4], where the authors considered the problem without the source term on the right-hand side and roughly speaking with the flux having jump discontinuities with respect to u and showed the existence and uniqueness result of a weak entropy solution. Later the framework was extended to fluxes discontinuous both in x and u in [3] and further in [9], where the formulation encounters also the source term on the right-hand side, but under some rather restrictive assumptions like continuity and monotonicity. The problem with a one-side Lipschitz source term (but with a continuous flux) was studied in [8] by the methods of set-valued analysis. However, in none of these works the kinetic formulation for such problems has been introduced. The existence proof of entropy weak solutions followed the scheme of regularizing the discontinuous flux and adapting in some way the ideas of Kružkov [10] and the framework of entropy measure valued solutions (DiPerna [6], Szepessy [15]). In the current paper we significantly relax the assumptions on the source term in comparison to [9] and we define the *kinetic formulation* and *the weak entropy solution* to (1.1) and show their equivalence. Finally, we present the constructive proof of existence of a kinetic solution. It means that instead of a standard approximation by a diffusion term we approximate the kinetic equation by an equation of the Boltzmann type. This procedure is usually referred to as the *kinetic approximation*.

Furthermore, we also incorporate the notion of *entropy measure valued solution*, which will be shown to be equivalent to the above ones and introduce and prove the stability of measure valued solutions with respect to data, which will be the key tool for the uniqueness and the independence of solutions of parametrization.

The techniques presented here apply to a more general setting, in particular to the fluxes and sources which in addition are x-discontinuous in a way that they satisfy the structural assumptions introduced in [1,12] and further essentially generalized in [3], where however the discontinuous source term is not taken into account.