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Beyond the Melnikov method: A computer assisted approach

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Abstract

We present a Melnikov type approach for establishing transversal intersections of stable/unstable manifolds of perturbed normally hyperbolic invariant manifolds (NHIMs). The method is based on a new geometric proof of the normally hyperbolic invariant manifold theorem, which establishes the existence of a NHIM, together with its associated invariant manifolds and bounds on their first and second derivatives. We do not need to know the explicit formulas for the homoclinic orbits prior to the perturbation. We also do not need to compute any integrals along such homoclinics. All needed bounds are established using rigorous computer assisted numerics. Lastly, and most importantly, the method establishes intersections for an explicit range of parameters, and not only for perturbations that are 'small enough', as is the case in the classical Melnikov approach.

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1. Introduction

The presence of the transversal intersection between stable and unstable manifolds for fixed point or periodic orbit is one of the main technical tools used to prove the chaotic behavior of the deterministic dynamical system (see for example [16] and the literature given there). In the context of the small perturbations of an integrable system the basic analytical technique used to establish the transversality is the Melnikov method [23] introduced in 1963. V.I. Arnold generalized these ideas to produce the first example of what is now called Arnold Diffusion [1]. In fact the (now widely-used) Melnikov function (see for example [28,19]) is, up to a constant, exactly the integral that Poincaré derived from Hamilton–Jacobi theory to obtain his obstruction to integrability in the restricted three body problem in [27].

Melnikov type methods are based on investigating integrals along homoclinic orbits to normally hyperbolic invariant manifolds (NHIMs) [9,11,12,19,23,28]. There are natural problems with such approach: It is very rarely the case that one can establish analytic formulae for such homoclinics. In most cases they are not known, and then computing integrals along them is impossible. The second problem is that even if one has an analytic formula for the homoclinic, the integral in question can be very hard to compute. In most real life systems such integrals would not be expressed through simple formulas.

One way to overcome such problems, adopted in [18], is by using exponential dichotomy and rigorous numerics. This evolves using approximate homoclinics, error bounds, and computer validated Newton type operator.

We resolve these two problems in a different way. Firstly, we investigate the dependence of the manifolds on the parameter using geometric and computer assisted tools. The slopes of the manifolds depending on the parameter follow from cone condition type bounds in the state space extended by the parameters. Second order derivatives also follow from geometric structures. This way we obtain bounds on the stable and unstable manifolds of NHIMs, together with their dependence up to second order on the perturbation parameter. We then propagate these bounds using rigorous (interval based) integration up to a section where they meet. Based on the bounds, and in particular using the dependence on the manifolds on the perturbation parameter, we establish transversal intersections for a given, explicit, range of perturbations. The range is large enough so that for the larger parameters from the range we can detect the transversal intersections directly, and continue to higher perturbations using standard techniques.

Our contribution to the existing theory is twofold:

Firstly, in this paper we develop a method for establishing center unstable manifolds of NHIMs, in the context of ordinary differential equations. The main benefit from our approach is that we do not need to assume that the NHIM exists in order to apply our method. (Our method is constructive, not perturbative.) We formulate assumptions, which guarantee the existence of a center-unstable manifold within an investigated neighborhood. The assumptions of our theorem depend only on the bounds on the first derivative of the vector field. These guarantee that the center-unstable manifold exists, and is a graph of a function within the investigated region. The method gives explicit bounds on the slope of the manifold. Moreover, by considering bounds on the second derivative of the vector field, we obtain explicit estimates on the second order derivatives of the center-unstable manifold. By changing the sign of the vector field, the method establishes existence of center-stable manifolds. By intersecting the center-stable manifold with the center-unstable manifold we establish the existence of a NHIM within the investigated region. Our method also establishes bounds on the first and second order dependence of the manifolds on