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Existence of ground state solutions to Dirac equations with vanishing potentials at infinity

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Abstract

In this work we study the existence of ground-state solutions of Dirac equations with potentials which are allowed to vanish at infinity. The approach is based on minimization of the energy functional over a generalized Nehari set. Some conditions on the potentials are given in order to overcome the lack of compactness.

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1. Introduction

In 1928 Dirac proposed a model to the quantum mechanics which, in contrast to the Schrödinger theory, takes into account the Relativity Theory. More specifically, he proposed a model to describe the evolution of a free relativistic particle, given by

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$$i\hbar \frac{\partial \psi}{\partial t} = D_c \psi, \tag{1.1}$$

where the operator D_c is given by

$$D_c = -ic\hbar \boldsymbol{\alpha} \cdot \nabla + mc^2\beta = -ic\hbar \sum_{k=1}^{3} \alpha_k \partial_k + mc^2\beta$$

and $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$, β satisfy the anticomutation relations

$$\begin{cases} \alpha_k\alpha_l + \alpha_l\alpha_k &= 2\delta_{kl} I, \\ \alpha_k\beta + \beta\alpha_k &= 0, \\ \beta^2 &= I, \end{cases} \tag{1.2}$$

where I denotes the identity matrix. It can be proved that the least dimension where (1.2) can hold is $N = 4$, where α_i and β are four-dimensional complex matrices given by

$$\beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \alpha_k = \begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix},$$

for $k = 1, 2, 3$ and σ_k given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Hence, the operator D_c is well defined in $L^2(\mathbb{R}^3, \mathbb{C}^4)$.

Let us consider the nonlinear Dirac equation

$$-i\hbar \frac{\partial \psi}{\partial t} = ic\hbar \sum_{k=1}^{3} \alpha_k \partial_k \psi - mc^2\beta\psi - M(x)\psi + g(x, \psi). \tag{1.3}$$

Assuming that $g(x, e^{i\theta}\psi) = g(x, \psi)$, by the *Ansatz* $\psi(t, x) = e^{\frac{i\mu t}{\hbar}} u(x)$, one can verify that $\psi(x, t)$ satisfies (1.3) if and only if $u : \mathbb{R}^3 \to \mathbb{C}^4$ satisfies the following problem

$$-i\hbar \sum_{k=1}^{3} \alpha_k \partial_k u + a\beta u + V(x)u = f(x, u), \tag{1.4}$$

where $a = mc^2$, $V(x) = M(x)/c + \mu I_4$ and $f(x, u) = g(x, u)/c$.

There are many works dedicated to study the Dirac equation (1.4) with the potential V and the nonlinearity f under several different hypotheses. In [11], Merle study the problem (1.4) with a constant potential $V(x) = \omega \in (-a, a)$ and nonlinearity representing the so called Soler model. As far as variational methods are concerned, it seems that Esteban and Séré in [10] were pioneers in using this kind of method to study (1.4).

Motivated by the versatility that variational methods provide, and by the physical appeal of its deduction, some researchers started to work in several generalizations of results which was