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Initial–boundary value problems for the general coupled nonlinear Schrödinger equation on the interval via the Fokas method ☆

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Abstract

Boundary value problems for integrable nonlinear differential equations can be analyzed via the Fokas method. In this paper, this method is employed in order to study initial–boundary value problems of the general coupled nonlinear Schrödinger equation formulated on the finite interval with 3×3 Lax pairs. The solution can be written in terms of the solution of a 3×3 Riemann–Hilbert problem. The relevant jump matrices are explicitly expressed in terms of the three matrix-value spectral functions $s(k)$, $S(k)$, and $S_L(k)$. The associated general Dirichlet to Neumann map is also analyzed via the *global relation*. It is interesting that the relevant formulas can be reduced to the analogous formulas derived for boundary value problems formulated on the half-line in the limit when the length of the interval tends to infinity. It is shown that the formulas characterizing the Dirichlet to Neumann map coincide with the analogous formulas obtained via a Gelfand–Levitan–Marchenko representation.

MSC: 41A60; 35Q51; 35Q15

Keywords: Riemann–Hilbert problem; Integrable system; Coupled nonlinear Schrödinger equation; Initial–boundary value problem; Dirichlet-to-Neumann map

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1. Introduction

Integrable partial differential equations (PDEs) have the defining property that they are equivalent to two linear eigenvalue equations, *called a Lax pair* [1]. Initial value problems for integrable evolution equations on the line can be analyzed via an effective method, the *so-called inverse scattering transform* (IST) [2,3]. In this case, the time evolution of the associated Lax spectral functions is simple, and the solution at time t can be recovered via the solution of an inverse problem. This inverse problem is most conveniently formulated as a Riemann–Hilbert (RH) problem whose jump matrix involves the given spectral functions.

In 1997, Fokas [4] (also see [5–7]) using ideas of IST introduced a new method, the *so-called Fokas method*, for analyzing boundary value problems for linear and for integrable nonlinear PDEs. For integrable nonlinear PDEs, the Fokas method is based on the simultaneous spectral analysis of the Lax pair, as well as on the analysis of an algebraic relation coupling the initial conditions with all boundary values, which is called by Fokas as the *global relation*.

In the past 19 years, using the Fokas method, boundary value problems of several important integrable equations with 2×2 Lax pair equations have been investigated, including the sine-Gordon [8], the Korteweg–de Vries [9], the nonlinear Schrödinger equations [10–12], etc. [13–29]. The Fokas method provides an expression for the solution of an IBV problem in terms of the solution of a Riemann–Hilbert problem. Particularly, an effective way analyzing the asymptotic behavior of the solution is based on this Riemann–Hilbert problem and on the nonlinear version of the steepest descent method introduced by Deift and Zhou [30].

In 2012, Lenells implemented the Fokas method to IBV problems for integrable evolution equations with Lax pair equations involving 3×3 matrices on the half-line [31]. There are several novelties for the transition from 2×2 to 3×3 matrix Lax pair equations. On the other hand, the two main steps of the method are similar to the analogous ones of [4,7]:

(I) Using a matrix Riemann–Hilbert formulated on the complex k-sphere, construct an integral representation of the solution, where k is the spectral parameter of the Lax pair equations. In general, some unknown boundary values are included in this representation, thus the formula of the solution is not yet effective.

(II) Considering the so-called global relation, analyze the unknown boundary values. In general, the representation of the unknown boundary values involves the solution of a nonlinear problem.

Following Lenells' work, the IBV problems of some important integrable evolution equations with 3×3 Lax pair have been investigated, including the Degasperis–Procesi [32], Sasa–Satsuma [33,34], three wave [35], the two-component nonlinear Schrödinger equations [36–38]. However, to the best of authors' knowledge, there are very few studies on the IBV problems for integrable equations with 3×3 matrices Lax pair on the finite interval. It is worth mentioning that in [36], Xu and Fan implemented the Fokas method to IBV problems for the Manokov equation on the Interval. The Manokov equation is the two-component nonlinear Schrödinger equation, which can be considered as a special case for the GCNLSS (1.1) by taking $a = c$ and $b = 0$.

The purpose of this paper is to study the IBV problems of the general coupled nonlinear Schrödinger equation (1.1) on the interval with a Lax pair equation involving 3×3 matrices. There exist some difficulties for solving IBV problems for 3×3 Lax pair equations on the interval. In order to implement step (I) above, four integration curves should be provided from