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Global existence and finite time blow-up of solutions of a Gierer–Meinhardt system ☆

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Abstract

We are concerned with the Gierer–Meinhardt system with zero Neumann boundary condition:

$$\begin{cases} u_t = d_1\Delta u - a_1 u + \frac{u^p}{v^q} + \delta_1(x), & x \in \Omega,\ t > 0, \\ v_t = d_2\Delta v - a_2 v + \frac{u^r}{v^s} + \delta_2(x), & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases}$$

where $p > 1$, $s > -1$, q, r, d_1, d_2, a_1, a_2 are positive constants, δ_1, δ_2, u_0, v_0 are nonnegative smooth functions, $\Omega \subset \mathbb{R}^d$ $(d \geq 1)$ is a bounded smooth domain. We obtain new sufficient conditions for global existence and finite time blow-up of solutions, especially in the critical exponent cases: $p - 1 = r$ and $qr = (p-1)(s+1)$.

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1. Introduction

In this paper, of our concern is the following general Gierer–Meinhardt system

$$\begin{cases} u_t = d_1\Delta u - a_1 u + \frac{u^p}{v^q} + \delta_1(x), & x \in \Omega,\ t > 0, \\ v_t = d_2\Delta v - a_2 v + \frac{u^r}{v^s} + \delta_2(x), & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $p > 1$, $s > -1$, q, r, d_1, d_2, a_1, a_2 are positive constants, δ_1, δ_2, u_0, v_0 are nonnegative smooth functions, $\Omega \subset \mathbb{R}^d$ is a bounded domain with smooth boundary $\partial\Omega$ and the space dimension $d \geq 1$, and ν is the unit outward norm vector on $\partial\Omega$.

Following the idea of *diffusion-driven instability* proposed by Turing [42], Gierer and Meinhardt [13] in 1972 introduced the reaction–diffusion system (1.1) to model the pattern formation of spatial tissue structures of hydra in morphogenesis, a biological phenomenon which was discovered by Trembley in 1744 [43]. It is noted that in the original Gierer–Meinhardt model, δ_1 is a nonnegative constant and $\delta_2 \equiv 0$; the general Gierer–Meinhardt model (1.1) was proposed in [15].

The Gierer–Meinhardt system (1.1) is one of the most famous models in biological pattern formation and belongs to the activator–inhibitor type. In the past few decades, the Gierer–Meinhardt system (1.1) has received extensive attention in research. The existence and uniqueness of a local solution to (1.1) is a folklore fact of standard parabolic theory; see, for example, [28], for details. Throughout the paper, a solution of (1.1) always means a classical nonnegative one. From the pure mathematical point of view, a fundamental question is the global existence of solution to (1.1). By a global solution of (1.1) we mean that its maximum existence time $T_{max} = \infty$, and by a blow-up solution (u, v) we mean that its maximum existence time $T_{max} < \infty$ and $\lim_{t \to T_{max}^-} \sup_{x\in\Omega}(u(x,t) + v(x,t)) = \infty$.

In the paper, unless otherwise stated, we assume that the initial data (u_0, v_0) satisfy

$$u_0(x) \geq 0, \quad v_0(x) > 0, \quad \forall x \in \overline{\Omega}.$$

According to the strong maximum principle for parabolic equations, the solution (u, v) of (1.1) satisfies $v(x,t) > 0$ for all $x \in \overline{\Omega}$, $0 < t < T_{max}$, and if either $u_0 \geq, \not\equiv 0$ or $\delta_1 \not\equiv 0$, then $u(x,t) > 0$ for all $x \in \overline{\Omega}$, $0 < t < T_{max}$.

The global existence and finite time blow-up of solution to the Gierer–Meinhardt system (1.1) have been studied extensively, for instance, in [1,10,15,17,26–28,38,49–51]. In what follows, let us briefly recall the existing results in this regard. First, in the special case $d \leq 3$, $p = 2$, $r = 2$, $q = 1$, $s = 0$ (so $p - 1 < r$ and $qr > (p-1)(s+1)$), the global existence of solution of (1.1) with δ_1 a nonnegative constant and $\delta_2 \equiv 0$ was established by Rothe [38] in 1984. Later in 1987, Masuda and Takahashi [28] improved the global existence result to the case $\frac{p-1}{r} < \frac{d}{d+2}$, $qr > (p-1)(s+1)$ under the extra condition that δ_1 is a positive constant and $\delta_2 \equiv 0$.

In [49], when δ_1 is a nonnegative constant and $\delta_2 \equiv 0$, for any given $p > 1$, $s > -1$, $q, r > 0$, by constructing a pair of suitable super-subsolutions, Wu and Li proved that