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J. Differential Equations 262 (2017) 615–632

Journal of Differential Equations

www.elsevier.com/locate/jde

Local L_∞-estimates, weak Harnack inequality, and stochastic continuity of solutions of SPDEs

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Received 10 March 2016; revised 13 September 2016

Available online 5 October 2016

Abstract

We consider stochastic partial differential equations under minimal assumptions: the coefficients are merely bounded and measurable and satisfy the stochastic parabolicity condition. In particular, the diffusion term is allowed to be scaling-critical. We derive local supremum estimates with a stochastic adaptation of De Giorgi's iteration and establish a weak Harnack inequality for the solutions. The latter is then used to obtain pointwise almost sure continuity.

MSC: 60H15; 60G46

Keywords: Stochastic PDEs; Harnack's inequality; De Giorgi iteration

1. Introduction

Harnack inequalities, introduced by [8], provide a comparison of values at different points of nonnegative functions which satisfy a partial differential equation (PDE). Inequalities of this type have a vast number of applications, in particular, they played a significant role in the study of PDEs with discontinuous coefficients in divergence form. This is the celebrated De Giorgi–Nash–Moser theory [6,20,18], in which Hölder continuity of the solutions is established. Later,

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http://dx.doi.org/10.1016/j.jde.2016.09.038
0022-0396/

by using a weaker version of Harnack's inequality, a simpler proof in the parabolic case was given in [15]. Harnack inequality and Hölder estimate for equations in non-divergence form, also known as the Krylov–Safonov estimate, was proved in [14] and [24]. Since then, similar results have been proved for more general equations, including for example integro-differential operators of Lévy type (see [2]) and singular equations (see [7] and references therein).

It is well known (see e.g. [13,12]) that the stochastic partial differential equations (SPDEs)

$$du_t = L_t u_t\,dt + M_t^k u_t\,dw_t^k, \tag{1.1}$$

where M^k are first order differential operators, are in many ways the natural stochastic extensions of parabolic equations $du_t = L_t u_t\,dt$. It is therefore also natural to ask whether the above mentioned results, fundamental in deterministic PDE theory, have stochastic counterparts. That is, what properties can one obtain for weak solutions of (1.1) without posing any smoothness assumptions on the coefficients? Note also that with bounded coefficients the diffusion term in (1.1) is critical to the parabolic scaling, and hence the question above fits in the recent activity in parabolic regularity with critical lower order terms, see e.g. [3] and its references. In some recent works regularity results have been obtained, but only for equations with at most zero order M, that is, with subcritical noise, for variants of this problem we refer to [10,9,5], and [16]. The methods in all of these works rely strongly on the absence of the derivatives in the noise, in which case the difficulty coming from the lack of regularity of the coefficients can be separated from the stochastic nature of the equation and can be essentially reduced to the deterministic case. In particular, adaptation of the classical techniques of [6,20,18] to the stochastic setting is not required, which is indeed what the scaling heuristic would suggest.

Concerning equations of the general form and under minimal assumptions – boundedness, measurability, and ellipticity – on the coefficients, few results are known. They were considered in [4] (see also [22]) and, in a backward setting, in [21], where global boundedness of the solutions was proved. In the present paper, we prove local L_∞-estimates for certain functions of the solutions, in terms of the corresponding L_2-norms, by using a stochastic version of De Giorgi's iteration. By virtue of these estimates, following the approach of [15], we establish a stochastic version of the aforementioned weak Harnack inequality in Theorem 2.2. Here "weak" stands for that in order to estimate the minimum of a nonnegative solution u, not only the maximum of u is required to be bounded from below by 1, but u itself on a positive portion of the domain. For deterministic equations by elementary arguments one can deduce Hölder continuity from such a weak Harnack inequality. These considerations however are quite sensitive to the measurability problems arising with the presence of stochastic terms, and therefore we need a far less straightforward argument to prove stochastic continuity of the solutions, which is formulated in Theorem 2.3. We note that Harnack inequalities for solutions of SPDEs – not to be confused with Harnack inequalities for the transition semigroup of SPDEs, for which we refer the reader to [25] and the references therein – have not been previously established even for equations with smooth coefficients.

Let us introduce the notations used throughout the paper. Let $d \geq 1$, and for $R \geq 0$ let $B_R = \{x \in \mathbb{R}^d : |x| < R\}$, $G_R = [4 - R^2, 4] \times B_R$, and $G := G_2$. $\mathcal{B}(B_R)$ will denote the Borel σ-algebra on B_R. Subsets of $\mathbb{R}^{d+1}$ of the form $J \times (B_R + x)$, where J is a closed interval in $[0, 4]$ and $x \in \mathbb{R}^d$, will be referred to as cylinders. If A is a set, I_A will denote the indicator function of A. The inner product in $L_2(B_2)$ will be denoted by $(\cdot, \cdot)$. The set of all compactly supported smooth functions on B_R will be denoted by $C_c^\infty(B_R)$. The space of $L_2(B_R)$-functions whose generalized derivatives of first order lie in $L_2(B_R)$ is denoted by $H^1(B_R)$, while the completion