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A method for obtaining time-periodic L^p estimates

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Abstract

We introduce a method for showing *a priori* L^p estimates for time-periodic, linear, partial differential equations set in a variety of domains such as the whole space, the half space and bounded domains. The method is generic and can be applied to a wide range of problems. We demonstrate it on the heat equation. The main idea is to replace the time axis with a torus in order to reformulate the problem on a locally compact abelian group and to employ Fourier analysis on this group. As a by-product, maximal L^p regularity for the corresponding initial-value problem follows *without* the notion of $\mathcal{R}$-boundedness. Moreover, we introduce the concept of a time-periodic fundamental solution.
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1. Introduction

The purpose of this article is to introduce a generic method that establishes L^p estimates for time-periodic solutions to a large class of linear, partial differential equations. The method works particularly well for parabolic problems. To emphasize the strength and simplicity of the method in the parabolic case, we demonstrate it on the time-periodic heat equation:

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$$\begin{cases} \partial_t u - \Delta u = f & \text{in } \mathbb{R}\times\Omega, \\ u = 0 & \text{on } \mathbb{R}\times\partial\Omega, \\ u(t+T,x) = u(t,x), & \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$ is a domain, $u : \mathbb{R}\times\Omega \to \mathbb{R}$ the unknown function dependent on a time variable t and a spatial variable x with $(t,x) \in \mathbb{R}\times\Omega$, $T > 0$ a fixed positive time-period, and $f : \mathbb{R}\times\Omega \to \mathbb{R}$ the data, which is also assumed to be T-time-periodic. The method to be introduced enables us to identify a Banach space X^p of T-time-periodic functions vanishing on the boundary $\partial\Omega$ with the property that the differential operator $\partial_t - \Delta$ maps X^p homeomorphically onto $L^p\big((0,T)\times\Omega\big)$ for any $p \in (1,\infty)$. As a consequence, we obtain the L^p estimate

$$\|u\|_{X^p} \leq c\,\|f\|_{L^p((0,T)\times\Omega)} \tag{1.2}$$

for a solution to (1.1). Observe that $L^p\big((0,T)\times\Omega\big)$ is the "natural" L^p space for T-time-periodic data, whence X^p can be described as the maximal regularity space in the L^p setting for the T-time-periodic problem (1.1). The Banach space X^p will be characterized as a Sobolev-type space. A proper identification of X^p, and the L^p estimate (1.2), is particularly important in the investigation of nonlinear problems; see also Remark 2.8.

Most investigations of time-periodic problems rely on results for the corresponding initial-value problems. For example, a Poincaré map is often employed to establish existence of at least one initial value that produces a time-periodic solution; see for example [10]. In contrast, the method to be introduced here is much more direct and does not rely on results for the corresponding initial-value problems at all. In particular, maximal regularity results for the initial-value problem are not needed. In fact, we shall show that maximal L^p regularity for the corresponding initial-value problem follows from our method effortlessly and *without* the use of $\mathcal{R}$-boundedness. Recall that a family of operators $\mathcal{T} \subset \mathcal{L}(\mathcal{X})$ on a Banach space $\mathcal{X}$ is called $\mathcal{R}$-bounded, if there exists a constant $c > 0$ such that

$$\int_0^1 \Big\| \sum_{j=1}^n \varepsilon_j(t) T_j x_j \Big\|_{\mathcal{X}} \,\mathrm{d}t \leq c \int_0^1 \Big\| \sum_{j=1}^n \varepsilon_j(t) x_j \Big\|_{\mathcal{X}} \,\mathrm{d}t$$

for all $n \in \mathbb{N}$, all $T_1,\ldots,T_n \in \mathcal{T}$, all $x_1,\ldots,x_n \in \mathcal{X}$, and all symmetric, independent, $\{-1,1\}$-valued random variables $\varepsilon_1,\ldots,\varepsilon_n$ on $[0,1]$. As our method is essentially based on estimates of Fourier multipliers, whereas the verification of $\mathcal{R}$-boundedness goes far beyond such estimates, this underlines the strength of our approach.

We shall first treat the whole-space case $\Omega = \mathbb{R}^n$, then the half-space case $\Omega = \mathbb{R}^n_+$ and finally the case of a sufficiently smooth bounded domain. In the first case, we establish a direct representation formula for the solution u in terms of Fourier multipliers. The estimate (1.2) is then shown using classical tools from harmonic analysis. In the half space case, the reflection principle applies. For the case of bounded domains, we employ localization techniques. One may recognize these steps as the standard procedure for analyzing elliptic problems. In fact, we consider it a novelty of our method that it enables us to treat time-periodic parabolic problems with the same approach that is typically used to investigate the corresponding elliptic problem.

We start by briefly explaining the method in the whole-space case $\Omega = \mathbb{R}^n$. The main idea is to reformulate the time-periodic problem as a PDE on the locally compact abelian group $G := \mathbb{T}\times\mathbb{R}^n$, where $\mathbb{T}$ denotes the torus $\mathbb{R}/T\mathbb{Z}$. As both the data f and the solution u are T-periodic

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