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Inverse source problem for the hyperbolic equation with a time-dependent principal part

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Abstract

In this paper, we investigate the inverse problem on determining the spatial component of the source term in the hyperbolic equation with a time-dependent principal part. Based on a Carleman estimate for general hyperbolic operators, we prove a local stability result of Hölder type in both cases of partial boundary and interior observation data. Numerically, we adopt the classical Tikhonov regularization to reformulate the inverse problem into a related optimization problem, for which we develop an iterative thresholding algorithm by using the corresponding adjoint system. Numerical examples up to three spatial dimensions are presented to demonstrate the accuracy and efficiency of the proposed algorithm.

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1. Introduction

Let $\Omega \subset \mathbb{R}^n$ ($n = 1, 2, \ldots$) be an open bounded domain with a smooth boundary $\partial\Omega$ (e.g., of C^2-class), and let $\nu = \nu(x) = (\nu_1(x), \ldots, \nu_n(x))$ be the outward unit normal vector to $\partial\Omega$ at $x \in \partial\Omega$. For some $T > 0$, set $Q := \Omega \times (-T, T)$. We consider the following initial value problem for a hyperbolic equation whose principal part depends on the time variable

$$\begin{cases} (\partial_t^2 - \mathcal{A}(t))u(x,t) = F(x,t) & ((x,t) \in Q), & (1.1) \\ u(x,0) = \partial_t u(x,0) = 0 & (x \in \Omega), & (1.2) \end{cases}$$

where

$$\begin{aligned} \mathcal{A}(t)u(x,t) &:= \operatorname{div}(a(x,t)\nabla u(x,t)) + b(x,t) \cdot \nabla u(x,t) + c(x,t)u(x,t) \\ &= \sum_{i,j=1}^{n} \partial_i(a_{ij}(x,t)\partial_j u(x,t)) + \sum_{i=1}^{n} b_i(x,t)\partial_i u(x,t) + c(x,t)u(x,t). \end{aligned}$$

Here $a = (a_{ij})_{1 \le i,j \le n}$ is a symmetric matrix, $b = (b_i)_{1 \le i \le n}$ is a vector, and there exists a constant $\kappa > 0$ such that

$$a(x,t)\xi \cdot \xi = \sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \ge \kappa|\xi|^2 = \kappa \sum_{i=1}^{n} \xi_i^2 \quad (\forall (x,t) \in \overline{Q},\ \forall \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n).$$

The regularities of a, b, c, the assumptions on the source term F and the boundary condition will be specified later. We denote the normal derivative associated with the elliptic operator $\mathcal{A}(t)$ as

$$\partial_{\mathcal{A}} u := a\nu \cdot \nabla u \quad \text{on } \partial\Omega \times (-T, T).$$

The well-posedness result concerning (1.1) will be provided in Lemma 2.1.

The main focuses of this paper are the theoretical stability and the numerical treatment for the following inverse source problem.

Problem 1.1. Let a subboundary $\Gamma \subset \partial\Omega$, a subdomain $\omega \subset \Omega$ and $T > 0$ be suitably given. Assume that the source term $F(x,t) = f(x)R(x,t)$ in (1.1) where R is given, and let u satisfy (1.1)–(1.2). Determine $f(x)$ by

Case (I) the partial boundary observation data $\{u, \partial_{\mathcal{A}} u\}|_{\Gamma \times (-T,T)}$, or

Case (II) the partial interior observation data $u|_{\omega \times (-T,T)}$.

Investigating the above problem from both theoretical and numerical aspects not only originates from the interest in mathematics, but also roots in its significance in practice. In the formulation of Problem 1.1, the source term $f(x)R(x,t)$ is incompletely separated into its spatial and temporal components, and we attempt to determine the spatial component f. Especially, if the source term is in the form of complete separation of variables, i.e. R is space-independent, (1.1) becomes an approximation to a model for elastic waves, and the term $f(x)R(t)$ acts as the