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Classification of isolated singularities of positive solutions for Choquard equations

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Abstract

In this paper we classify the isolated singularities of positive solutions to Choquard equation

$$-\Delta u + u = I_\alpha[u^p]u^q \ \text{ in } \mathbb{R}^N \setminus \{0\}, \quad \lim_{|x|\to+\infty} u(x) = 0,$$

where $p > 0, q \geq 1, N \geq 3, \alpha \in (0, N)$ and $I_\alpha[u^p](x) = \int_{\mathbb{R}^N} \frac{u(y)^p}{|x-y|^{N-\alpha}} dy$. We show that any positive solution u is a solution of

$$-\Delta u + u = I_\alpha[u^p]u^q + k\delta_0 \quad \text{in } \mathbb{R}^N$$

in the distributional sense for some $k \geq 0$, where δ_0 is the Dirac mass at the origin. We prove the existence of singular solutions in the subcritical case: $p + q < \frac{N+\alpha}{N-2}$ and $p < \frac{N}{N-2}, q < \frac{N}{N-2}$ and prove that either the solution u has removable singularity at the origin or satisfies $\lim_{|x|\to 0^+} u(x)|x|^{N-2} = C_N$ which is a positive constant. In the supercritical case: $p + q \geq \frac{N+\alpha}{N-2}$ or $p \geq \frac{N}{N-2}$, or $q \geq \frac{N}{N-2}$ we prove that $k = 0$.

MSC: 35J75; 35B40

Keywords: Classification of singularity; Choquard equation; Nonlocal problem; Decay asymptotic

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1. Introduction

In this paper, we are concerned with the classification of all positive solutions to Choquard equation

$$\begin{aligned} -\Delta u + u = I_\alpha[u^p]u^q \quad &\text{in} \quad \mathbb{R}^N \setminus \{0\}, \\ \lim_{|x|\to+\infty} u(x) = 0, & \end{aligned} \tag{1.1}$$

where $p > 0$, $q \geq 1$, $N \geq 3$, $\alpha \in (0, N)$ and

$$I_\alpha[u^p](x) = \int_{\mathbb{R}^N} \frac{u(y)^p}{|x-y|^{N-\alpha}}\, dy.$$

When $N = 3$, $\alpha = p = 2$ and $q = 1$, problem (1.1) was proposed by P. Choquard as an approximation to Hartree–Fock theory for a one component plasma, which has been explained in Lieb and Lieb–Simon's papers [20,21] respectively. It is also called Choquard–Pekar equation after a more early work of S. Paker for describing the quantum mechanics of a polaron at rest [31], or sometime the nonlinear Schrödinger–Newton equation in the context of self-gravitating matter [34]. The Choquard type equations also arise in the physics of multiple-particle systems, see [19]. Furthermore, the Choquard type equations appear to be a prototype of the nonlocal problems, which play a fundamental role in some Quantum-mechanical and non-linear optics, refer to [18,30]. When $\alpha \in (0, 2)$, the Riesz potential I_α is related to the fractional Laplacian, which is a nonlocal operator, so the Choquard equation (1.1) could be divided into a system with the Laplacian in the linear part of the first equation and fractional Laplacian in the second one. For the related topics on the fractional equation we can refer for example to [7–11,24].

The study of isolated singularities is initiated by Brezis and Lions in [5], where an useful tool to connect the singular solutions of elliptic equation in punctured domain and the solutions of corresponding elliptic equation in the distributional sense was built, by the study of

$$\begin{aligned} &\Delta u \leq au + f \quad \text{in } \Omega \setminus \{0\}, \quad u > 0 \ \text{ in } \Omega \setminus \{0\}, \\ &u \in L^1(\Omega), \quad \Delta u \in L^1(\Omega \setminus \{0\}), \end{aligned}$$

where Ω is a bounded domain in $\mathbb{R}^N$ containing the origin, the parameter $a > 0$ and function $f \in L^1(\Omega)$. Later on, the classification of isolated singular problem

$$\begin{aligned} -\Delta u = u^p \quad &\text{in} \quad \Omega \setminus \{0\}, \\ u > 0 \quad &\text{in} \ \ \Omega \end{aligned} \tag{1.2}$$

was performed by Lions in [22] for $p \in (1, \frac{N}{N-2})$, by Aviles in [1] for $p = \frac{N}{N-2}$, by Gidas–Spruck in [16] for $\frac{N}{N-2} < p < \frac{N+2}{N-2}$, by Caffarelli–Gidas–Spruck in [6] for $p = \frac{N+2}{N-2}$. For the case that $p > \frac{N+2}{N-2}$, the classification of isolated singularities for (1.2) is still open. In the particular case of $p \in (1, \frac{N}{N-2})$, any positive solution of (1.2) is a solution of

$$-\Delta u = u^p + k\delta_0 \quad \text{in} \quad \Omega \tag{1.3}$$