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$C^{1+\alpha}$-strict solutions and wellposedness of abstract differential equations with state dependent delay

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Abstract

We study the existence and uniqueness of $C^{1+\alpha}$-strict solutions for a general class of abstract differential equations with state dependent delay. We also study the local well-posedness of this type of problems on subspaces of $C^{1+\alpha}([-p, 0]; X)$. Some examples involving partial differential equations are presented.

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1. Introduction

The main objective of this paper is to study the existence and uniqueness of strict and $C^{1+\alpha}$-strict solutions for a general class of abstract functional differential equations with state dependent delay of the form

$$u'(t) = Au(t) + F(t, u_{\sigma(t,u_t)}), \quad t \in [0, a], \tag{1}$$

$$u_0 = \varphi \in \mathcal{B}_X = C([-p, 0]; X), \tag{2}$$

where $A : D(A) \subset X \to X$ is the generator of an analytic semigroup of bounded linear operators $(T(t))_{t\geq 0}$ defined on a Banach space $(X, \|\cdot\|)$ and $F(\cdot), \sigma(\cdot)$ are continuous functions to be specified later.

The literature on differential equations with state dependent delay is recent and extensive. For ordinary differential equations we cite the survey by Hartung, Krisztin, Walther and Wu [2], the early work by Aiello, Freedman and Wu [1], the works of Hartung [3], Hartung et al. [4,5], H-O. Walther [23] and the references therein. Concerning abstract differential equations and partial differential equations, we mention Rezounenko [16–18], Hernández et al. [7,8] and Wu et al. [10,14,20]. We also cite the recent papers Krisztin & Rezounenko [9] and Yunfei, Yuan & Pei [12], where the existence and smoothness of semiflows of C^1-solutions on a submanifold of $C([-r, 0]; X)$ is studied for state-dependent delay partial differential equations of parabolic type of the form $u'(t) = Au(t) + f(u(t - r(u_t)))$, where A is the generator of an analytic semigroup on $L^2(\Omega)$. The results in [9,12] are established under the assumption that $f(\psi) = g(\psi(-r(\psi)))$ where $g(\cdot)$ is a bounded linear operator from $L^2(\Omega)$ into a fractional space X_α. It is an interesting and elegant condition, but is also an important restriction.

In this work we study the existence and uniqueness of strict and $C^{1+\alpha}$-strict solutions for (1)–(2). In particular, by assuming that $F : [0, a] \times X \to X$ is smooth and that $F(\cdot)$, $D_2F(\cdot)$ are α-Hölder, we establish the existence and uniqueness of $C^{1+\alpha}$-strict solutions and we prove that the problem (1)–(2) is locally well posed (in a suitable sense) on the subspace of $C^{1+\alpha}([-p, 0]; X)$ given by $\mathcal{S}_{X,\alpha} = \{\varphi \in C^{1+\alpha}([-p, 0]; X) : \varphi(0) \in D(A), \varphi'(0) = A\varphi(0) + F(0, \varphi) \in \mathcal{D}_{\alpha,\infty}\}$, where $\mathcal{D}_{\alpha,\infty}$ is an interpolation space.

We include now some brief comments of our results. It is well known that the problem (1)–(2) is not well posed in the usual space $C([-p, 0]; X)$, which is a consequence of the fact that mapping as $u \to F(t, u_{\sigma(t,u_t)})$ is not Lipschitz in the usual space of continuous functions. However, estimates of the form

$$\begin{aligned} &\| F(t, u_{\sigma(t,u_t)}) - F(t, v_{\sigma(t,v_t)}) \| \\ &\quad \leq L_F(\|u - v\|_{C([0,b];X)} + [v]_{C_{Lip}([0,b];X)}[\sigma]_{C_{Lip}([0,b]\times\mathcal{B};\mathbb{R})} \| u - v\|_{C([0,b];X)}) \end{aligned} \tag{3}$$

(L_F is the Lipschitz constant of $F(\cdot)$) are available, which permit the study of the existence and uniqueness of strict solution when $\varphi \in C_{Lip}([-p, 0]; X)$. By using the inequality (3), in Theorem 2 and Proposition 1 we prove the existence and uniqueness of a strict solution for the abstract problem (1)–(2). These results are formulated in a very general way, permitting diverse applications, see in particular Corollary 1 and Corollary 2.

To study the existence and uniqueness of $C^{1+\alpha}$ strict solutions via fixed point techniques and the semigroup theory, we need estimate terms of the form $[F(\cdot, u_{\sigma(\cdot,u_{(\cdot)})}) - F(\cdot, v_{\sigma(\cdot,v_{(\cdot)})})]_{C^\alpha([0,b];X)}$. To this end, in Proposition 2 we prove non-trivial estimates for