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Regularity results for nonlinear parabolic obstacle problems with subquadratic growth

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Abstract

In [26] it was shown that the spatial gradient of the solution u to the parabolic obstacle problem with superquadratic growth is local Hölder continuous provided the obstacle is regular enough. In this paper, we extend this regularity result to the subquadratic case. This means we establish the local Hölder continuity of the spatial gradient of the solution u to the parabolic obstacle problem with subquadratic growth. More precisely, we prove that

$$Du \in C_{\mathrm{loc}}^{0;\alpha,\frac{\alpha}{2}} \quad \text{for some } \alpha \in (0,1),$$

provided the coefficients and the obstacle are regular enough. Moreover, we use the local Hölder continuity to prove the local Lipschitz continuity of the solution u, i.e.

$$u \in C_{\mathrm{loc}}^{0;1,\frac{1}{2}}.$$

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1. Introduction

The aim of this paper is to establish the local Hölder continuity of the spatial gradient of the solution u to the parabolic obstacle problem in the subquadratic case, i.e. the growth exponent satisfies $\frac{2n}{n+2} < p < 2$ with $n \geq 2$. Here, we will extend the result shown in [26] for the case $p \geq 2$ and thereby, we will complete the theory of the local Hölder continuity of the spatial gradient of the solution u to the parabolic obstacle problem. This result we will also use to prove that u is local Lipschitz continuous.

In general, obstacle problems are interesting objects in the theory of partial differential equations and the calculus of variations. The obstacle problem is a classic motivating example in the mathematical study of variational inequalities and free boundary problems. The theory of obstacle problems is also motivated by numerous applications, e.g. in mathematical physics, in mechanics, in control theory or in mathematical biology. We refer to [1,33] for an overview of the classical theory and applications. Up to now, the theory for elliptic problems is well understood, as well as the theory for elliptic obstacle problems. However, the case of non-linear parabolic problems with general obstacle functions remained open for a long time. First results were achieved by Bögelein, Duzaar and Mingione [5] and then, by Scheven [41]. Here, we want to highlight that in [5] the authors established the first existence result to parabolic problems with irregular obstacles, which are not necessarily non-increasing in time. They consider general obstacles with the only additional assumption that the time derivative of the obstacle lies in $L^{p'}$. This is required since their method relies on a time mollification argument, combined with a maximum construction in order to recover the obstacle condition, where the pointwise maximum construction is not compatible with distributional time derivatives. Moreover, they established the Calderón–Zygmund theory for a large class of parabolic obstacle problems, i.e. they proved that the (spatial) gradient of solutions is as integrable as that of the assigned obstacles. Then, in [41] Scheven introduced a new concept of solution to parabolic obstacle problems of p-Laplacian type with highly irregular obstacles, the so-called *localizable solutions*, see Definition 1. The main feature of localizable solutions is that they solve the obstacle problem locally, which is a priori not clear by the formulation of the problem, cf. the remarks preceding Definition 1. This new concept allows to consider more general settings, i.e. it is no more necessary to assume that the time derivative of the obstacle function lies in $L^{p'}$. It suffices to consider obstacles with distributional time derivatives. Moreover, we want to emphasize that the concept of localizable solutions allows to prove more general regularity results. Scheven also proved Calderón–Zygmund estimates for parabolic obstacle problems. The main difference between the result of Scheven and the result of Bögelein, Duzaar and Mingione is that in [5] they need an additional assumption on the boundary data, which seems to be unnatural for proving regularity in the interior. The reason for the additional assumption on the boundary data arises from the fact that the formulation of the obstacle problem is not of local nature. Bögelein, Duzaar and Mingione used a complex approximation argument to approximate the solutions by more regular ones and since the given solution was not known to be localizable, this approximation procedure had to be implemented on the whole domain. This problem could be avoid by the concept of localizable solutions.

These existence results enable also many regularity results for parabolic problems with irregular obstacle, see e.g. [2,8,6,26]. In the context of higher integrability of solutions, we have to mention a further result, which is given by Bögelein and Scheven in [6]. They proved the self-improving property of integrability for parabolic obstacle problems without any monotonicity