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Lifespan estimates for the semi-linear Klein–Gordon equation with a quadratic potential in dimension one ☆

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Abstract

We show for almost every $m > 0$, the solution to the semi-linear Klein–Gordon equation with a quadratic potential in dimension one, exists over a longer time interval than the one given by local existence theory, using the normal form method. By using an $L^p - L^q$ estimate for eigenfunctions of the harmonic oscillator and by carefully analysis on the nonlinearity, we improve the result obtained by the author before.

MSC: 35L70; 35J15

Keywords: Harmonic oscillator; Klein–Gordon equations; Long time existence

1. Introduction and the main result

We are concerned with lower bounds for the lifespan of the solution to the semi-linear Klein–Gordon equation with a quadratic potential in dimension $d = 1$:

$$\begin{cases} (\partial_t^2 - \Delta + |x|^2 + m^2)v = v^{p+1}, & (t,x) \in [-T,T] \times \mathbb{R}^d \\ v|_{t=0} = \epsilon f, \quad \partial_t v|_{t=0} = \epsilon g \end{cases} \tag{1.1}$$

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with $m > 0$, $p \in \mathbb{N}^*$, $\epsilon > 0$ small enough and $(f, g) \in \mathscr{H}^{s+1}(\mathbb{R}^d) \times \mathscr{H}^s(\mathbb{R}^d)$, where for a natural number s

$$\mathscr{H}^s(\mathbb{R}^d) = \{v \in L^2(\mathbb{R}^d) : x^\alpha \partial^\beta v \in L^2(\mathbb{R}^d), \forall \alpha, \beta \in \mathbb{N}^d \text{ s.t. } |\alpha| + |\beta| \leq s\}. \tag{1.2}$$

The similar equation without the quadratic potential $|x|^2$, and with small data, smooth and compactly supported, has almost global solutions when $d = 1$ (see Moriyama et al. [7]), and has global solutions in higher dimensions (see Delort et al. [3] and references therein). Their proofs rely on the use of the dispersive property of the linear solution. The situation is dramatically different when we replace $-\Delta$ by $-\Delta + |x|^2$. Since the operator $-\Delta + |x|^2$ has pure point spectrum, it turns out that solutions of the corresponding linear equation do not decay in time, and thus no dispersive property is available. Because of that the long time existence problem of (1.1) is similar to the corresponding problem on compact manifolds.

Let us mention some known results about lower bound lifespan estimates for $(\partial_t^2 - \Delta_g + m^2)v = v^{p+1}$ on compact manifolds with smooth small enough data of size ϵ, where Δ_g is the Laplace–Beltrami operator. In the case of spheres $\mathbb{S}^d (d \geq 1)$ (or more generally Zoll manifolds), the solution exists, for almost every $m > 0$, on a time interval of length $c_N \epsilon^{-N}$ for any $N \in \mathbb{N}$ (see [1] and references therein). An important property is that the gap between eigenvalues of $\sqrt{-\Delta_g}$ on $\mathbb{S}^d$ is separated by a fixed constant. In the case of the torus $\mathbb{T}^d (d \geq 2)$, such a gap condition does not hold. Actually, two successive eigenvalues λ, λ' of $\sqrt{-\Delta_g}$ on $\mathbb{T}^d$ might be separated by an interval of length as small as c/λ, which means the gap shrinks to zero as eigenvalues tend to infinity. However, it has been proved that for almost every $m > 0$, the solution of such an equation exists over an interval of time of length $c\epsilon^{-p(1+2/d)}$ (up to a logarithm) and has Sobolev norms of high index bounded on such an interval (see [2]). Such a result was improved by Fang and the author, who showed that the solution exists over on a time interval of length almost of size $c\epsilon^{-\frac{3}{2}p}$ (slightly better than that in [2] when $d > 4$), under the same assumptions (see [4]).

Since the eigenvalues of $\sqrt{-\Delta + |x|^2}$ on $\mathbb{R}^d$ share the similar gap condition as in the case of the torus, one is able to get a lower bound of the time existence of order $c\epsilon^{-4p/3}$ when $d \geq 2$ and of order $c\epsilon^{-25p/18}$ when $d = 1$. A natural question is whether one could get a lower bound estimate for the lifespan of solutions to (1.1) as good as in the case of the torus. We give a positive answer to this question in dimension $d = 1$. The proof is based on the normal form method, which was first introduced for partial differential equations by J. Shatah in 1985 (see [8]).

Let us also mention that recently, a lower bound estimate of order $c\epsilon^{-3/2}$ when dimension $d \geq 3$ is odd and of order $c\epsilon^{-19/12}$ when $d = 1$, for the lifespan of the solution to (1.1) with $m = 0$ and $p = 1$, has been obtained (see [10]).

Let $\mathscr{H}^{s+1}(\mathbb{R}^d) \times \mathscr{H}^s(\mathbb{R}^d)$ be endowed with the norm $\|(f, g)\|_{\mathscr{H}^{s+1}(\mathbb{R}^d)\times\mathscr{H}^s(\mathbb{R}^d)} = \|f\|_{\mathscr{H}^{s+1}} + \|g\|_{\mathscr{H}^s}$. By local existence theory, problem (1.1) admits a unique solution defined on the time interval $|t| \leq c\epsilon^{-p}$ for any (f, g) in the unit ball of $\mathscr{H}^{s+1}(\mathbb{R}^d) \times \mathscr{H}^s(\mathbb{R}^d)$, provided s is large enough and $\epsilon > 0$ is small enough. The main result of this paper is the following:

Theorem 1.1. *Let $d = 1$. For any $\rho > 0$, there exists a zero measure subset $\mathscr{N}$ of $(0, \infty)$ such that for any $m \in (0, \infty) - \mathscr{N}$, there are $\epsilon_0 > 0, c > 0, s_0 > 0$ satisfying the following: for any natural number $s \geq s_0$, any $\epsilon \in (0, \epsilon_0)$, any real-valued function pair (f, g) belonging to the unit ball of $\mathscr{H}^{s+1}(\mathbb{R}) \times \mathscr{H}^s(\mathbb{R})$, problem (1.1) has a unique solution*

$$v \in C^0((-T_\epsilon, T_\epsilon), \mathscr{H}^{s+1}(\mathbb{R})) \cap C^1((-T_\epsilon, T_\epsilon), \mathscr{H}^s(\mathbb{R}))$$