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On the classification of elliptic foliations induced by real quadratic fields with center

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Abstract

Related to the study of Hilbert's infinitesimal problem, is the problem of determining the existence and estimating the number of limit cycles of the linear perturbation of Hamiltonian fields. A classification of the elliptic foliations in the projective plane induced by the fields obtained by quadratic fields with center was already studied by several authors.

In this work, we devise a unified proof of the classification of elliptic foliations induced by quadratic fields with center. This technique involves using a formula due to Cerveau & Lins Neto to calculate the genus of the generic fiber of a first integral of foliations of these kinds.

Furthermore, we show that these foliations induce several examples of linear families of foliations which are not bimeromorphically equivalent to certain remarkable examples given by Lins Neto.

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1. Introduction

The *infinitesimal's Hilbert Problem* asks for an upper bound to the number of limit cycles of a polinomial vector field of degree n, close to a polinomial vector field with first integral f. Even the case $n = 2$ is an open problem. In this case, there is some progress when f has elliptic curves as generic level curves (called *elliptic fibrations*) [1–4].

Any quadratic differential equation, for which the origin is a non-degenerated singularity of center type, can be taken to the following form

$$x' = y + a_{2,0}x^2 + a_{1,1}xy + a_{0,2}y^2,$$
$$y' = -x + b_{2,0}x^2 + b_{1,1}xy + b_{0,2}y^2.$$

We can also complexify the previous equation, to obtain

$$z' = -iz + Az^2 + Bz\bar{z} + C\bar{z}^2, \tag{1.1}$$

where $A, B, C \in \mathbb{C}$.

The integrability theory of Darboux [5] made it possible to obtain necessary and sufficient conditions for the classification theorem of centers of quadratic polynomial differential systems. The conditions for the existence of a analytic first integral of a quadratic center were due to Dulac and Kapteyn [6–8].

Theorem *(Dulac–Kapteyn). There are five types of quadratic systems with center:*

Q_3^H: $z' = -iz - z^2 + 2z\bar{z} + C\bar{z}^2$, $C \in \mathbb{C} \setminus \mathbb{R}$, *(Hamiltonian);*
H_Δ: $z' = -iz + \bar{z}^2$, *(Hamiltonian triangle);*
Q_3^{LV}: $z' = -iz + z^2 + C\bar{z}^2$, $C \in \mathbb{C}$, *(generalized Lotka–Volterra);*
Q_3^R: $z' = -iaz + 4z^2 + 2z\bar{z} + c\bar{z}^2$, $a, c \in \mathbb{R}$, *(Reversible);*
Q_4: $z' = -iz + 4z^2 + 2z\bar{z} + C\bar{z}^2$, $|C| = 2$, $C \in \mathbb{C} \setminus \mathbb{R}$, *(Codimension four).*

From Dulac–Kapteyn's theorem, we obtain the classification of quadratic vector fields with a center, namely, if the complex ODE (1.1) possesses a center then it must have a first integral of one of the following forms

$$P_3 \in \mathbb{R}[x, y], \quad \text{(Hamiltonian cases: } Q_3^H \text{ and Hamiltonian triangle);} \tag{1.2}$$

$$x^p y^q (ax + by + c)^r, \quad p, q \in \mathbb{Z},\ a, b, c \in \mathbb{R}, \quad \text{(Lotka–Volterra case: } Q_3^{LV}\text{);} \tag{1.3}$$

$$x^p (y^2 + P_2(x))^q, \quad q \in \mathbb{N},\ p \in \mathbb{Z},\ P_2 \in \mathbb{R}_2[x, y], \quad \text{(Reversible case: } Q_3^R\text{);} \tag{1.4}$$

$$\frac{P_3(x, y)^2}{P_2(x, y)^3}, \quad P_2, P_3 \in \mathbb{R}[x, y], \quad \text{(Codimension four case: } Q_4\text{).} \tag{1.5}$$

The classification of elliptic foliations was previously addressed by Iliev, Gautier, Gavrilov, among others, see [1,9–11] and the references therein. In particular, in [1], Gautier provides the classification of reversible and Lotka–Volterra foliations up to birational maps. Later on, in [9], Gautier et al. computed the *generating functions* (also called, *Poincaré–Pontryagin–Melnikov*