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J. Differential Equations 261 (2016) 4789–4824

Journal of Differential Equations

www.elsevier.com/locate/jde

Large time behavior of solutions for hyperbolic balance laws

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Received 8 March 2016

Available online 18 August 2016

Abstract

We study the existence and the large time behavior of global solutions to the initial value problem for hyperbolic balance laws in n space dimensions with $n \geq 3$ admitting an entropy and satisfying the stable condition. We first construct global existence of the solutions to such a system around a steady state if the initial energy is small enough. Then we show that k-order derivatives of these solutions approach a constant state in the L^p-norm at a rate $O(t^{-\frac{1}{2}(k+\rho+\frac{n}{2}-\frac{n}{p})})$ with $p \in [2, \infty]$ and $\rho \in [0, \frac{n}{2}]$ provided that initially $\|z_0\|_{\dot{B}^{-\rho}_{2,\infty}} < \infty$, where $\dot{B}^{-\rho}_{2,\infty}$ is a homogeneous Besov space. These decay results do not impose an additional smallness assumption on L^p norm of the initial data and we thus improve the results in [3,19]. We also show faster decay results in the sense that if $\|\mathbf{P}z_0\|_{\dot{B}^{-\rho}_{2,\infty}} + \|(\mathbf{I}-\mathbf{P})z_0\|_{\dot{B}^{-\rho+1}_{2,\infty}} < \infty$ with $\rho \in (\frac{n}{2}, \frac{n+2}{2}]$, k-order derivatives of the solutions approach a constant state in the L^p-norm at a rate $O(t^{-\frac{1}{2}(k+\rho+1+\frac{n}{2}-\frac{n}{p})})$.

1. Introduction

This work concerns the following N-component hyperbolic system of balance laws in n space dimensions:

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http://dx.doi.org/10.1016/j.jde.2016.07.016

0022-0396/

$$w_t + \sum_{j=1}^{n} f^j(w)_{x_j} = g(w), \tag{1.1}$$

with the initial conditions

$$w(0,x) = w_0(x), \tag{1.2}$$

where $t \geq 0$ and $x \in \mathbf{R}^n$ with $n \geq 3$, $w = w(t,x) \in \mathcal{O}_w \subset \mathbf{R}^N$ with $N \geq 1$ and $\mathcal{O}_w$ is open and convex. Here $f^j(w)$ $(j = 1, ..., n)$ and $g(w)$ are given N-vector valued smooth functions of w from $\mathcal{O}_w$ into $\mathbf{R}^N$. We define

$$\bar{\mathcal{N}} = \{\psi \in \mathbf{R}^N; \psi^T g(w) = 0 \text{ for any } w \in \mathcal{O}_w\}.$$

Here the superscript T denotes the transpose. We set the dimension of $\bar{\mathcal{N}}$ to be N_1 with $1 \leq N_1 \leq N-1$ and $\bar{\mathcal{N}}^\perp$ to be the orthogonal complement of $\bar{\mathcal{N}}$. Then we have $g(w) \in \bar{\mathcal{N}}^\perp$.

Such nonlinear systems typically govern non-equilibrium processes in physics for media with hyperbolic response as, for example, in gas dynamics. They also arise in the numerical simulation of conservation laws by relaxation schemes, see [1,3,4,8,12,37] and the references cited therein.

To obtain the existence of solutions to (1.1) and (1.2) the entropy condition is needed as in [9,10,18,37]. Here we first give the definition of entropy [18,19].

Definition 1.1. Let $\eta = \eta(w)$ be a smooth function defined in $\mathcal{O}_w$. Then $\eta(w)$ is called an entropy of balance laws (1.1) if the following four statements hold:

(a) $\eta(w)$ is strictly convex in $\mathcal{O}_w$ in the sense that the Hessian $D_w^2\eta(w)$ is positive definite for $w \in \mathcal{O}_w$.

(b) The matrix $D_w f^j(w)(D_w^2\eta(w))^{-1}$ is symmetric for $w \in \mathcal{O}_w$ and $j = 1, ..., n$.

(c) Let $w \in \mathcal{O}_w$. Then $g(w) = 0$ holds if and only if $(D_w\eta(w))^T \in \bar{\mathcal{N}}$.

(d) For $w \in \mathcal{O}_w$ satisfying $g(w) = 0$, the matrix $D_w g(w)(D_w^2\eta(w))^{-1}$ is symmetric and non-positive definite and its null space coincides with $\bar{\mathcal{N}}$.

Let $\eta = \eta(w)$ be defined as above and put $u(w) = (D_w\eta(w))^T$. It was shown in [18] that the mapping $u = u(w)$ is a diffeomorphism from $\mathcal{O}_w$ onto its range $\mathcal{O}_u$. Suppose that $w = w(u)$ is the inverse mapping from $\mathcal{O}_u$ onto $\mathcal{O}_w$. Putting $w = w(u)$ in (1.1) we have

$$\bar{A}^0(u)u_t + \sum_{j=1}^{n} \bar{A}^j(u)u_{x_j} = H(u). \tag{1.3}$$

Here

$$\bar{A}^0(u) := D_u w(u), \quad H(u) := g(w(u)), \tag{1.4}$$

$$\bar{A}^j(u) := D_u f_j(w(u)) = D_w f^j(w(u)) D_u w(u). \tag{1.5}$$

In addition we see that $H(u) \in \bar{\mathcal{N}}^\perp$ for any $u \in \mathcal{O}_u$. Let us define

$$\bar{L}(u) := -D_u H(u) = -(D_w g)(w(u)) D_u w(u). \tag{1.6}$$