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Ground states for irregular and indefinite superlinear Schrödinger equations ☆

Nils Ackermann *, Julián Chagoya

Instituto de Matemáticas, Universidad Nacional Autónoma de México, Circuito Exterior, C.U., 04510 México D.F., Mexico

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Abstract

We consider the existence of a ground state for the subcritical stationary semilinear Schrödinger equation $-\Delta u + u = a(x)|u|^{p-2}u$ in H^1, where $a \in L^\infty(\mathbb{R}^N)$ may change sign. Our focus is on the case where loss of compactness occurs at the ground state energy. By providing a new variant of the Splitting Lemma we do not need to assume the existence of a limit problem at infinity, be it in the form of a pointwise limit for a as $|x| \to \infty$ or of asymptotic periodicity. That is, our problem may be *irregular* at infinity. In addition, we allow a to change sign near infinity, a case that has never been treated before.

MSC: 35J61; 35J20

Keywords: Stationary Schrödinger equation; Ground state; Indefinite superlinear; Subcritical; No limit problem

1. Introduction

We are concerned with the subcritical stationary semilinear Schrödinger equation

$$-\Delta u + u = a(x)|u|^{p-2}u, \qquad u \in H^1(\mathbb{R}^N), \tag{1.1}$$

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* Corresponding author.
E-mail addresses: nils@ackermath.info (N. Ackermann), julian.chagoya@gmail.com (J. Chagoya).

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where $H^1 := H^1(\mathbb{R}^N)$ is the usual Sobolev space and $a \in L^\infty(\mathbb{R}^N)$. Here and in what follows function spaces are over $\mathbb{R}^N$ unless otherwise noted. Suppose throughout that $2 < p < 2^*$, where $2^* := 2N/(N-2)$ if $N \geq 3$, $2^* := \infty$ if $N = 1$ or 2, is the critical Sobolev exponent.

Solutions to the more general problem

$$-\Delta u + V(x)u = a(x)|u|^{p-2}u, \qquad u \in H^1(\mathbb{R}^N), \tag{1.2}$$

give rise to certain solitary waves of the corresponding time dependent Schrödinger or Klein–Gordon equations, and have therefore received much attention in the literature. Under appropriate conditions on V and a, weak solutions of (1.2) are in correspondence with the critical points of the variational functional (the "energy") $J\colon H^1 \to \mathbb{R}$ defined by

$$J(u) := \frac{1}{2}\int (|\nabla u|^2 + V|u|^2) - \frac{1}{p}\int a|u|^p.$$

A *ground state* of (1.2) is a minimum of J on the Nehari manifold

$$\{u \in H^1\backslash\{0\} \mid \mathrm{D}J(u)u = 0\},$$

which is a nontrivial critical point of J under suitable conditions on V and a. The problem of existence of ground states for (1.2) is of particular interest since they potentially yield orbitally stable standing wave solutions to the Schrödinger Equation [1,2]. The main obstacle to prove existence of solutions for (1.2) is the inherent lack of compactness, that is, the failure of the Palais–Smale or Cerami conditions for J due to the noncompact embedding $H^1 \hookrightarrow L^p$ for $p \in (2, 2^*)$.

If V and a are constant then existence of ground states of (1.1) was analyzed in the seminal work of Berestycki and Lions [3], see also the references therein. For the nonautonomous equation, existence of ground states has been considered under various hypotheses to overcome the lack of compactness.

If V and a are radially symmetric then compactness is restored in the radially symmetric subspace H^1_r of H^1 [4,5]. Depending on other properties of V and a a ground state in H^1_r may or may not be a ground state in H^1. If, roughly speaking, $\lim_{|x|\to\infty} V(x) = \infty$, $\limsup_{|x|\to\infty} a(x) \leq 0$ or $a^+ \in L^q$ for a suitable $q > 0$ then compactness is restored in H^1 or in an appropriately weighted space [6–19]. And, last but not least, replacing the right hand side of (1.2) by $f(x, u)$, where f is asymptotically linear in u, a nonresonance condition ensures compactness [20,21].

Apart from these cases, most results impose the existence of a limit problem at infinity and employ concentration compactness arguments. This can be achieved by assuming the existence of pointwise limits of V and a as $|x| \to \infty$, see, e.g., [4,12,22–37] or, more generally, [38–40]. Another variant of this approach is to assume (asymptotic) periodicity of V and a in the coordinates of the x variable, see, e.g., [15,30,41–48].

The only existence result for (1.2) in the setting without compactness we are aware of that does not impose a limit on V and a as $|x| \to \infty$ is [49]. Here the existence of a ground state is shown for $a \equiv 1$ and $\operatorname{ess\,inf} V > 0$, assuming that V takes values below $\liminf_{|x|\to\infty} V$ on a large enough ball (Theorem 1.2 in [49]). The condition is not explicit though and cannot be checked directly. Nevertheless, under explicit conditions on V the authors prove in Theorem 1.3 in [49] the existence of a solution, which is not a ground state.