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Self-adjoint elliptic operators with boundary conditions on not closed hypersurfaces

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Abstract

The theory of self-adjoint extensions of symmetric operators is used to construct self-adjoint realizations of a second-order elliptic differential operator on $\mathbb{R}^n$ with linear boundary conditions on (a relatively open part of) a compact hypersurface. Our approach allows to obtain Kreĭn-like resolvent formulae where the reference operator coincides with the "free" operator with domain $H^2(\mathbb{R}^n)$; this provides an useful tool for the scattering problem from a hypersurface. Concrete examples of this construction are developed in connection with the standard boundary conditions, Dirichlet, Neumann, Robin, δ and δ'-type, assigned either on a $(n-1)$ dimensional compact boundary $\Gamma = \partial\Omega$ or on a relatively open part $\Sigma \subset \Gamma$. Schatten–von Neumann estimates for the difference of the powers of resolvents of the free and the perturbed operators are also proven; these give existence and completeness of the wave operators of the associated scattering systems.

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1. Introduction

This work is concerned with the self-adjoint realizations of symmetric, second-order elliptic operators

$$Au(x) = \sum_{1\le i,j\le n} \partial_{x_i}(a_{ij}(x)\partial_{x_j}u(x)) - V(x)u(x)\,, \quad x \equiv (x_1,\dots,x_n) \in \mathbb{R}^n\,, \tag{1.1}$$

with boundary conditions on (relatively open parts of) hypersurfaces which are boundaries Γ of bounded open sets $\Omega \subset \mathbb{R}^n$. We assume Ω to be of class $\mathcal{C}^{1,1}$; that suffices in case the boundary conditions are globally imposed on Γ whereas, in the case of boundary conditions on $\Sigma \subset \Gamma$, we require more regularity on Γ, even if it suffices to assume Σ to be of class $\mathcal{C}^{1,0}$, i.e. Σ has a Lipschitz boundary. By [37,13], we expect that our results can be extended to the case in which also Γ is merely Lipschitz. As regards the conditions on the coefficients a_{ij} and V, for simplicity we assume that they are both in $\mathcal{C}^\infty_b(\mathbb{R}^n)$, the standard regularity hypotheses allowing to use the classical results on mapping properties of surface potentials (as given, for example, in [57, Chapter 6]). However our regularity assumptions could possibly be relaxed. For example, in the case $a_{ij} = \delta_{ij}$, it should suffice to work with any $(-\Delta)$-bounded potential V; moreover, by following the results concerning the surface potentials provided in [1] and references therein, we expect that our analysis could also be adapted to the case where the a_{ij}'s are bounded and Lipschitz and V is bounded.

When defined on the domain $\mathrm{dom}(A) = H^2(\mathbb{R}^n)$, the operator A is self-adjoint and bounded from above. We then consider the same differential operator A but now acting on a domain characterized by linear boundary conditions on Γ or on a relatively open part $\Sigma \subset \Gamma$. Using the abstract theory of self-adjoint extensions of symmetric operators developed in [60–63], we construct these models as singular perturbations of the "free operator" with domain $H^2(\mathbb{R}^n)$. This allows us to describe all possible linear boundary conditions within an unified framework where the corresponding self-adjoint operators $A_{\Pi,\Theta}$ are parametrized through couples (Π, Θ), where Π is an orthogonal projector on the Hilbert trace space $H^{\frac{3}{2}}(\Gamma) \oplus H^{\frac{1}{2}}(\Gamma)$ and Θ is a self-adjoint operator in the Hilbert space given by the range of Π. Our approach naturally yields to Kreĭn-type formulae expressing the resolvent of the self-adjoint extension $A_{\Pi,\Theta}$ in terms of the unperturbed resolvent $(-A+z)^{-1}$, plus a non-perturbative term; under suitable regularity assumptions of the parameters (Π, Θ), the difference $(-A_{\Pi,\Theta}+z)^{-k} - (-A+z)^{-k}$ is of trace class (for sufficiently large k) and the Birman–Kato criterion allows to consider $\{A, A_{\Pi,\Theta}\}$ as a scattering system provided with the corresponding wave operators.

Singular perturbations supported on manifolds of lower dimension have been the object of a large number of investigations (see for instance [2–5,8–12,14–16,21–33,35,36,42,51–54,59, 60] and references therein). These have mainly concerned the case of δ-perturbed Schrödinger operators and are generally motivated by the quantum dynamical modeling, as the case of leaky quantum graphs, or the quantum interaction with charged surfaces.

Covering a wider class of models, the analysis developed in our work have been inspired by the scattering problem from a compact hypersurface with abstract boundary conditions. When these conditions are encoded by the extension $A_{\Pi,\Theta}$, the scattered field u_{sc} corresponding to an incident wave u_{in} is expected to be related to a limit absorption principle for $\left(-A_{\Pi,\Theta}+z\right)^{-1}$. In particular, the result obtained in the simpler case of point scatterers (see [47]) suggests the relation