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An optimal estimate for electric fields on the shortest line segment between two spherical insulators in three dimensions

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Abstract

We consider a gradient estimate for a conductivity problem whose inclusions are two neighboring insulators in three dimensions. When inclusions with an extreme conductivity (insulators or perfect conductors) are closely located, the gradient can be concentrated in between inclusions and then becomes arbitrarily large as the distance between inclusions approaches zero. The gradient estimate in between insulators in three dimensions has been regarded as a challenging problem, while the optimal blow-up rates in terms of the distance were successfully obtained for the other extreme conductivity problems in two and three dimensions, and are attained on the shortest line segment between inclusions. In this paper, we establish upper and lower bounds of gradients on the shortest line segment between two insulating unit spheres in three dimensions. These bounds present the optimal blow-up rate of gradient on the line segment which is substantially different from the rates in the other problems.

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1. Introduction

Let B_1 and B_2 be bounded and simply connected domains in $\mathbb{R}^d$, $d = 2, 3$. We consider the following conductivity problem: for a given harmonic function H defined in $\mathbb{R}^d$,

$$\begin{cases} \nabla \cdot \big((k-1)\chi_{(B_1 \cup B_2)} + 1\big)\, \nabla u = 0 & \text{in } \mathbb{R}^d, \\ u(\mathbf{x}) - H(\mathbf{x}) = O(|\mathbf{x}|^{1-d}) & \text{as } |\mathbf{x}| \to \infty, \end{cases}$$

where χ is the characteristic function. Two inclusions B_1 and B_2 are conductors with conductivity $k \neq 1$, embedded in the background with conductivity 1. For an extreme conductivity $k = 0$ or ∞, a modified model has been used, see (2.1) and [22]. Let ϵ denote the distance between B_1 and B_2, i.e.,

$$\epsilon := \operatorname{dist}(B_1, B_2),$$

and we assume that the distance ϵ is small.

The problem is to estimate $|\nabla u|$ in the narrow region in between inclusions. This was raised by Babuška in relation to the study of material failure of composites [4]. In fiber-reinforced composites which consist of stiff fibrous inclusions and the matrix, high shear stress concentrations can occur in between closely spaced neighboring inclusions. It is important to estimate the shear stress tensor ∇u, while u means the out-of-plane displacement, and the inclusions B_1 and B_2 are the cross-sections of fibers. Many studies have been extensively conducted on the gradient estimate due to such practical importance.

Successful results have been achieved in all cases except three dimensional insulators which we consider in this paper. Such successful results can be divided into three cases when k stays away from 0 and ∞, when k degenerates to either 0 (insulating) or ∞ (perfectly conducting) in two dimensions, and when $k = \infty$ in three and higher dimensions. On the other hand, this paper deals with the exceptional case when $k = 0$ in three dimensions. We prove the occurrence of concentration in the narrow region, and also established the optimal blow-up rate for $|\nabla u|$ on the shortest line segment between two insulating unit spheres in terms of ϵ.

We give a brief description of three successful cases mentioned above. In the first case when k stays away from 0 and ∞, i.e., $c_1 < k < c_2$ for some positive constants c_1 and c_2, it was proved by Li–Vogelius [18] that $|\nabla u|$ remains bounded regardless of the distance ϵ between inclusions. The boundedness result was extended to elliptic systems by Li–Nirenberg [17].

In the second case when k is either 0 or ∞ in the two dimensional problem, the gradient ∇u can become unbounded as the distance ϵ tends to 0, and the generic blow-up rate of $|\nabla u|$ is $1/\sqrt{\epsilon}$. For two circular inclusions, the blow-up rate $1/\sqrt{\epsilon}$ was derived by Budiansky–Carrier [9], and Kang–Lim et al. [3,2] established the precise dependence of $|\nabla u|$ on ϵ, radii of disks and $k \in [0, \infty]$. For inclusions in a sufficiently general class of shapes in $\mathbb{R}^2$, it was shown by Yun [22,23] that the blow-up rate $1/\sqrt{\epsilon}$ is valid at $k = 0$ or ∞, see also [20] for an enhancement of concentration. Taking it a step further, an asymptotic for the distribution ∇u was established by Kang–Lim–Yun [11], when B_1 and B_2 are disks. For sufficiently general shapes of inclusions, recent results by Ammari et al. [1] and Kang–Lee–Yun [10] yield a numerically stable method to well describe the asymptotic behavior of ∇u in $\mathbb{R}^2$.

In the third case when $k = \infty$ in three and higher dimensions, Bao–Li–Yin [7] proved that the generic blow-up rate for the perfectly conducting inclusions is $|\epsilon \log \epsilon|^{-1}$ in three dimensions and $|\epsilon|^{-1}$ in higher ones, see also [8] for multiple inclusions. Lim–Yun [19] also found