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Non-uniformly hyperbolic flows and shadowing

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Abstract

We consider a hyperbolic ergodic measure of a C^1 flow on a compact manifold. Under the hypothesis that there are no fixed points and that the Oseledec splitting of the normal bundle satisfies a limit domination property, we prove that the measure has a shadowing property. As an application of this result we prove that the measure can be approached on the weak* topology by measures supported on hyperbolic periodic orbits.

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1. Introduction

Shadowing properties appear in dynamical systems in many forms and they are always very important, they are related to the existence and denseness of periodic orbits. These are old prob-

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lems in dynamical systems. Nowadays, questions about the growth of the number of periodic orbits, uniqueness and ergodicity of equilibrium states and the existence of horseshoes are all topics related to this subject. The literature covering these aspects is vast, see the book [15] and the bibliography therein.

Vaguely speaking, a shadowing property is the property that an approximate orbit with errors small enough has a true periodic orbit nearby (see the precise definition below). For this kind of property the system stays unchanged without any perturbation. Then it is natural that at least some kind of weak hyperbolicity is necessary in order to assure its verification. We just mention some seminal work in this direction and suggest the reader to have a look in the book [15] for more detailed treatment and references. Anosov [2] and Bowen [6–8], in their pioneering work, considered situations with uniformly hyperbolic behaviour. Katok [9] considered the case of non-uniformly hyperbolic diffeomorphisms. It is still under investigation whether some kind of non-uniformly hyperbolic flows have some kind of shadowing property. Indeed, in the case of flows there is no hyperbolicity along the orbits and re-parameterizations of trajectories become necessary and make the problem more difficult. We consider the situation of C^1 non-uniformly hyperbolic flows and prove that, under the assumption of limit domination, a non-uniformly hyperbolic C^1 flow without singularity has the shadowing property, this is the content of Theorem 1 below. This shadowing can be regarded as a generalization of uniformly hyperbolic case, see [7, (2.4) on p. 10].

The main results of this article are stated in the end of Section 2. This section begins with notation, concepts and some facts like the Oseledec Theorem for flows. Section 3 contains properties of limit dominated splittings, more concepts and the statement of an important shadowing lemma due to Liao. This section also introduces the Liao–Pesin set of a hyperbolic ergodic measure of a flow and states two theorems about this set. Section 4 contains the proofs of the main theorems.

2. Concepts and main results

Throughout this article M is a compact n-dimensional C^1 Riemannian manifold without boundary endowed with a distance ρ induced by the Riemannian metric, $\phi : M \times \mathbb{R} \to M$ is a C^1 flow without fixed points and $\phi_t : M \to M$ is the time t diffeomorphism generated by ϕ. We say that a set $B \subseteq M$ is ϕ invariant if $\phi_t(B) = B$, for every $t \in \mathbb{R}$, a Borel probability measure μ is ϕ invariant if $\mu(\phi_t(B)) = \mu(B)$, for every $t \in \mathbb{R}$ and every Borel set B. A ϕ invariant measure is called *ergodic* if $\mu(B) = 0$ or $\mu(B) = 1$, for every ϕ invariant Borel set B.

For $x \in M$, an orbit segment of x from time t_1 to t_2 is by definition the set

$$\mathrm{Orb}(x, [t_1, t_2]) = \{\phi_t(x) : t_1 \leq t \leq t_2\}.$$

If I is an interval containing the origin, a function $\sigma : I \to \mathbb{R}$ is called a *re-parametrization*, if it is continuous, strictly increasing and maps the origin on itself. We denote the set of all re-parameterizations on I by $\mathrm{Rep}(I)$.

We introduce below the notion of shadowing and the shadowing property which we are interested in. On the contrary to what happens for diffeomorphisms, in the context of flows, we need to allow re-parameterizations of the trajectories in order to adjust the speed and control distances and this causes new difficulties to be solved.