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Versal unfolding of planar Hamiltonian systems at fully degenerate equilibrium ☆

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Abstract

In this paper we study bifurcations of a planar Hamiltonian system at a fully degenerate equilibrium, which has a zero linearization. Since the Poincaré normal form theory is not applicable to such a degenerate system, we investigate its restrictive normal forms in the class of Hamiltonian fields and prove that such a degenerate system is of codimension 3 degeneracy in the class, so that we introduce three parameters to versally unfold the degenerate system in the class. In order to discuss further the qualitative properties of the versal unfolding, we use the Poincaré index to determine the number and distribution of hyperbolic sectors near the degenerate equilibrium. We display its all bifurcations such as pitchfork bifurcation, saddle-center bifurcation and the Bogdanov–Takens bifurcation within Hamiltonian systems.

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Keywords: Hamiltonian system; Versal unfolding; Normal form; Degeneracy; Bifurcation

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1. Introduction

As indicated in [6,10], an unfolding is usually a deformation with small unfolding parameters of a degenerate system in the topological space of all C^∞ vector fields. A versal unfolding means a family of vector fields whose local flows contain all possible small perturbations of the degenerate flow in the sense of equivalence of local families. There is no general recipe available for constructing such a family. Each case must be considered individually. One usually wants to find the simplest versal unfolding, called *universal unfolding*, which needs the least number of unfolding parameters. For this purpose one needs to find the simplest equivalent form for the general deformation of the degenerate system. The well-known Poincaré normal form theory [6,7,10] provides such a method for simplest equivalent forms. In 1936, Williamson [17] investigated normal forms for linear Hamiltonian systems. His idea is to transform the corresponding matrix of the linear part into the simplest form while keeping the symplectic structure of the system. For the nonlinear Hamiltonian systems, normal forms were investigated by Birkhoff [3], Cherry [5], Meer [12] and Siegel [15]. As shown in Section 2.7 of [7], a nonlinear Hamiltonian system can be reduced by a series of near identity symplectic transformations to a normal form such that its kth-order homogeneous terms are resonant polynomials of order k. However, this method works usually under the condition that the matrix of linear part is not zero. Otherwise, the fully zero linearization makes all higher order terms being resonant polynomials [6, pp. 402–420] or [7, pp. 122–142].

Based on the reduction to normal forms, there have been great number of works on universal unfoldings of planar Hamiltonian system since Aleksandr M. Liapunov published his celebrated center theorem (see e.g. Abraham and Marsden's [1, p. 498]) in 1895, which attracted attentions to deformations of Hamiltonian systems. It is worthy mentioning that the universal unfolding for quadratic with nilpotent linear part was given by Bogdanov [4] and Takens [16] in 1970's. Later, a versal unfolding of the Hamiltonian system near a cusp of order 3 was discussed by Dumortier, Roussarie and Sotomayor [8].

Some Hamiltonian systems contain a fully zero linear part, i.e., the origin is a fully degenerate equilibrium. For example, the system

$$\begin{cases} \dot{x} = axy + \epsilon x^3, \\ \dot{y} = by^2 + cx^2, \end{cases} \tag{1.1}$$

where a, b, c, ϵ are parameters and $a = -2b$, was investigated in Section 13.3 of [6]. When $\epsilon = 0$, system (1.1) is Hamiltonian and has a fully zero linearization at the equilibrium $O : (0, 0)$. Unlike the cusp case considered in [4,8,16], we cannot use the Poincaré normal form theory or results of Hamiltonian normal form to give its universal unfolding because the fully zero linearization makes all higher order terms be resonant and none of those terms can be eliminated by a near-identity transformation. Although it is hard to unfold such a system of higher degeneracy, we can change the routine of unfolding from all analytic vector fields to a part of them, for example, the vector fields of Hamiltonian form. Such a restriction means that only perturbations which preserve the Hamiltonian structure are considered. Therefore, we convert to investigate unfoldings of such a degenerate system in the class of Hamiltonian fields. However, the degeneracy of the system may change in the restricted class and a theory of normal forms in Hamiltonian form is needed.

In this paper we investigate the universal unfolding of a planar analytic Hamiltonian system at a fully degenerate equilibrium, which has a zero linearization. We give its versal unfolding