Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 261 (2016) 273–295

Journal of Differential Equations

www.elsevier.com/locate/jde

Periodic-parabolic eigenvalue problems with a large parameter and degeneration

Daniel Daners *, Christopher Thornett [1]

School of Mathematics and Statistics, University of Sydney, NSW 2006, Australia

Received 10 September 2015; revised 4 March 2016

Abstract

We consider a periodic-parabolic eigenvalue problem with a non-negative potential λm vanishing on a non-cylindrical domain D_m satisfying conditions similar to those for the parabolic maximum principle. We show that the limit as $\lambda \to \infty$ leads to a periodic-parabolic problem on D_m having a periodic-parabolic principal eigenvalue and eigenfunction which are unique in some sense. We substantially improve a result from [Du and Peng, Trans. Amer. Math. Soc. 364 (2012), p. 6039–6070]. At the same time we offer a different approach based on a periodic-parabolic initial boundary value problem. The results are motivated by an analysis of the asymptotic behaviour of positive solutions to semilinear logistic periodic-parabolic problems with temporal and spacial degeneracies.

MSC: 35K20; 35P15; 35B10

Keywords: Non-autonomous parabolic initial-boundary value problem; Periodic-parabolic eigenvalue problem; Limit problems for a large parameter; Spatial and temporal degeneration

* Corresponding author.
E-mail addresses: daniel.daners@sydney.edu.au (D. Daners), thornett@cims.nyu.edu (C. Thornett).
[1] Current address: Courant Institute of Mathematical Sciences, New York University, 251 Mercer Street, New York, NY 10012, USA.

http://dx.doi.org/10.1016/j.jde.2016.03.007
0022-0396/

1. Introduction

We consider a periodic-parabolic eigenvalue problem arising in the study of the asymptotic behaviour of positive solutions to a T-periodic logistic type population problem such as first studied in [27,7] and later in [2,3,21,22,24,33]. The limiting behaviour of the eigenvalue problem allows to deduce information about the corresponding logistic-type semilinear problem. Our focus is in on the case of temporal and spacial degeneracies motivated in particular in [22].

More precisely, we are interested in the behaviour of the principal eigenvalue for the periodic-parabolic eigenvalue problem

$$\begin{aligned} \frac{\partial u}{\partial t} + \mathcal{A}(t)u + \lambda m(x,t)u = \mu(\lambda)u \qquad & \text{in } \Omega \times (0,T), \\ \mathcal{B}(t)u = 0 \qquad & \text{in } \partial\Omega \times (0,T), \\ u(x,0) = u(x,T) \qquad & \text{in } \Omega, \end{aligned} \tag{1.1}$$

as $\lambda \to \infty$, where $m \in L^\infty\big(\Omega \times (0,T)\big)$ is a non-negative weight function that has a non-trivial zero set satisfying suitable assumptions. Moreover, $\Omega \subseteq \mathbb{R}^N$ is a bounded domain, and

$$\mathcal{A}(t)u := -\operatorname{div}\big(D(x,t)\nabla u + a(x,t)u\big) + \big(b(x,t)\cdot\nabla u + c_0(x,t)u\big) \tag{1.2}$$

is a uniformly strongly elliptic operator with bounded and measurable coefficients and $\mathcal{B}(t)$ a boundary operator of Dirichlet, Neumann or Robin type (for precise assumptions see Section 2).

As in [27], a principal eigenvalue of (1.1) is an eigenvalue having a *positive* eigenvector. If $m(x,t) > 0$ on $\Omega \times (0,T)$ nothing interesting happens, so we focus on the case where $m(x,t) = 0$ in some region $D_m \subseteq \Omega \times [0,T]$ of non-zero measure. Such problems have been looked at in particular for the corresponding elliptic problem in [2,10,33]. The most general weights m are considered in [1,22,23,34,35], where spacial and temporal degeneration is allowed. Our aim is to simplify and generalise some of these results using an alternative method and allowing fully non-autonomous operators $(\mathcal{A}(t), \mathcal{B}(t))$ including the principal part.

The approach we take is quite different from previous work and related to the one used in [17] for elliptic systems. Rather than studying the eigenvalue problem (1.1) directly we study what happens to the solution to

$$\begin{aligned} \frac{\partial u}{\partial t} + \mathcal{A}(t)u + \lambda m(x,t)u = 0 \qquad & \text{in } \Omega \times (s,T), \\ \mathcal{B}(t)u = 0 \qquad & \text{in } \partial\Omega \times (s,T), \\ u(x,s) = u_s(x) \qquad & \text{in } \Omega, \end{aligned} \tag{1.3}$$

as $\lambda \to \infty$, where $s \in [0,T)$. We consider the behaviour of weak solutions of (1.3) with a non-zero right hand side as $\lambda \to \infty$ in Section 2. In Section 3 we show that for every initial value $u_0 \in L^p(\Omega)$ the problem (1.3) has a unique solution $u \in C([s,T], L^p(\Omega))$. This in particular allows us to define the evolution operator $U_\lambda(t,s)$ by

$$U_\lambda(t,s)u_s := u(t). \tag{1.4}$$