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New existence and symmetry results for least energy positive solutions of Schrödinger systems with mixed competition and cooperation terms

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Abstract

In this paper we focus on existence and symmetry properties of solutions to the cubic Schrödinger system

$$-\Delta u_i + \lambda_i u_i = \sum_{j=1}^{d} \beta_{ij} u_j^2 u_i \quad \text{in } \Omega \subset \mathbb{R}^N, \qquad i = 1, \dots d$$

where $d \geqslant 2$, $\lambda_i, \beta_{ii} > 0$, $\beta_{ij} = \beta_{ji} \in \mathbb{R}$ for $j \neq i$, $N = 2, 3$. The underlying domain Ω is either bounded or the whole space, and $u_i \in H_0^1(\Omega)$ or $u_i \in H^1_{\text{rad}}(\mathbb{R}^N)$ respectively. We establish new existence and symmetry results for least energy positive solutions in the case of mixed cooperation and competition coefficients, as well as in the purely cooperative case.

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1. Introduction

The existence and the qualitative description of *least energy solutions* to the nonlinear elliptic system

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1 u^3 + \beta u v^2 \\ -\Delta v + \lambda_2 v = \mu_2 v^3 + \beta u^2 v \\ u, v \in H_0^1(\Omega), \end{cases} \qquad \text{with } \Omega \subset \mathbb{R}^N \text{ or } \Omega = \mathbb{R}^N, \text{ and } N = 2, 3, \tag{1.1}$$

have attracted considerable attention in the last ten years, starting from the seminal paper [15] by T.-C. Lin and J. Wei. Collecting all the results contained in several contributions, it is possible to obtain an exhaustive picture of the problem, see the forthcoming Subsection 1.1. In striking contrast, a complete understanding in the case of an arbitrary $d \geq 3$ components system

$$\begin{cases} -\Delta u_i + \lambda_i u_i = \sum_{j=1}^d \beta_{ij} u_j^2 u_i \quad \text{in } \Omega \\ u_i \not\equiv 0 \\ u_i \in H_0^1(\Omega), \end{cases} \qquad i = 1, \dots, d, \quad \beta_{ij} = \beta_{ji} \tag{1.2}$$

is not available, mainly due to the possible coexistence of *cooperation* and *competition*, that is, the existence of two pairs (i_1, j_1) and (i_2, j_2) such that $\beta_{i_1 j_1} > 0$ and $\beta_{i_2 j_2} < 0$. We recall that the sign of the coupling parameter β_{ij} determines the nature of the interaction between the components u_i and u_j: if $\beta_{ij} > 0$, then they cooperate, while if $\beta_{ij} < 0$, then they compete. Very recently, the systematic study of existence of least energy solutions in problems with simultaneous cooperation and competition has been started by the first author in [27] and by Y. Sato and Z.-Q. Wang in [22]. Nevertheless, there are still some gaps to fill in order to obtain a complete picture. In the present paper we give a contribution to fill some of these gaps, and, in the meantime, we analyse the symmetry properties of least energy solutions to (1.2), proving results which are new also in a purely cooperative context ($\beta_{ij} > 0$ for every $i \neq j$), and recover what is known in the purely competitive one ($d = 2$ and $\beta_{12} = \beta < 0$).

In order to motivate our research, in the following we review the results already available in the literature, but before it is worth to observe that thanks to the assumption $\beta_{ij} = \beta_{ji}$, system (1.2) has variational structure, as its solutions are critical points of the functional $J : H_0^1(\Omega, \mathbb{R}^d) \to \mathbb{R}$ defined by

$$J(\mathbf{u}) := \int_\Omega \frac{1}{2} \sum_{i=1}^d \left(|\nabla u_i|^2 + \lambda_i u_i^2 \right) - \int_\Omega \frac{1}{4} \sum_{i,j=1}^d \beta_{ij} u_i^2 u_j^2,$$

where we used the vector notation $\mathbf{u} = (u_1, \dots, u_d)$. Observe that (1.2) admits *semi-trivial solutions*, i.e., solutions $\mathbf{u} \not\equiv 0$ with some zero components. However, we will be only interested in the existence of *positive solutions*: $\mathbf{u}$ solving (1.2) such that $u_i > 0$ for every i. In particular, we will be interested in the existence of *least energy positive solutions*, that is solutions achieving the *least energy positive level*:

$$a := \inf \{ J(\mathbf{u}) : \ \mathbf{u} \text{ is a solution of (1.2) such that } u_i > 0 \text{ for all } i \},$$