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Existence of global weak solutions to compressible isentropic finitely extensible nonlinear bead–spring chain models for dilute polymers: The two-dimensional case

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Abstract

We prove the existence of global-in-time weak solutions to a general class of models that arise from the kinetic theory of dilute solutions of nonhomogeneous polymeric liquids, where the polymer molecules are idealized as bead–spring chains with finitely extensible nonlinear elastic (FENE) type spring potentials. The class of models under consideration involves the unsteady, compressible, isentropic, isothermal Navier–Stokes system in a bounded domain Ω in $\mathbb{R}^d$, $d = 2$, for the density ρ, the velocity $\underset{\sim}{u}$ and the pressure p of the fluid, with an equation of state of the form $p(\rho) = c_p \rho^\gamma$, where c_p is a positive constant and $\gamma > 1$. The right-hand side of the Navier–Stokes momentum equation includes an elastic extra-stress tensor, which is the classical Kramers expression. The elastic extra-stress tensor stems from the random movement of the polymer chains and is defined through the associated probability density function that satisfies a Fokker–Planck-type parabolic equation, a crucial feature of which is the presence of a centre-of-mass diffusion term. This extends the result in our paper J.W. Barrett and E. Süli (2016) [9], which established the existence of global-in-time weak solutions to the system for $d \in \{2, 3\}$ and $\gamma > \frac{3}{2}$, but the elastic extra-stress tensor required there the addition of a quadratic interaction term to the classical Kramers expression to complete the compactness argument on which the proof was based. We show here that in the case of $d = 2$ and $\gamma > 1$ the existence of global-in-time weak solutions can be proved in the absence of the quadratic interaction term. Our results require no structural assumptions on the drag term in the Fokker–Planck equation; in particular, the drag term need not be corotational. With a nonnegative initial density $\rho_0 \in L^\infty(\Omega)$ for the continuity

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equation; a square-integrable initial velocity datum $\underset{\sim}{u}_0$ for the Navier–Stokes momentum equation; and a nonnegative initial probability density function ψ_0 for the Fokker–Planck equation, which has finite relative entropy with respect to the Maxwellian M associated with the spring potential in the model, we prove, *via* a limiting procedure on a pressure regularization parameter, the existence of a global-in-time bounded-energy weak solution $t \mapsto (\rho(t), \underset{\sim}{u}(t), \psi(t))$ to the coupled Navier–Stokes–Fokker–Planck system, satisfying the initial condition $(\rho(0), \underset{\sim}{u}(0), \psi(0)) = (\rho_0, \underset{\sim}{u}_0, \psi_0)$.

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1. Introduction

In Barrett & Süli [9] we established the existence of global-in-time weak solutions to a large class of bead–spring chain models with finitely extensible nonlinear elastic (FENE) type spring potentials — a system of nonlinear partial differential equations that arises from the kinetic theory of dilute polymer solutions. We first restate the model considered there, and we then discuss the aims of the extensions of the results of [9] contained in the present paper.

In the model studied in [9], the solvent is a compressible, isentropic, viscous, isothermal Newtonian fluid confined to a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, with boundary $\partial\Omega$. For the sake of simplicity of presentation, Ω is assumed to have a 'solid boundary' $\partial\Omega$; the velocity field $\underset{\sim}{u}$ will then satisfy the no-slip boundary condition $\underset{\sim}{u} = \underset{\sim}{0}$ on $\partial\Omega$. The equations of continuity and balance of linear momentum have the form of the compressible Navier–Stokes equations (cf. Lions [18], Feireisl [14], Novotný & Straškraba [20], or Feireisl & Novotný [15]) in which the *elastic extra-stress* tensor $\underset{\approx}{\tau}$ (i.e., the polymeric part of the Cauchy stress tensor) appears as a source term in the conservation of momentum equation:

Given $T \in \mathbb{R}_{>0}$, find $\rho : (\underset{\sim}{x}, t) \in \Omega \times [0, T] \mapsto \rho(\underset{\sim}{x}, t) \in \mathbb{R}_{\geq 0}$ and $\underset{\sim}{u} : (\underset{\sim}{x}, t) \in \overline{\Omega} \times [0, T] \mapsto \underset{\sim}{u}(\underset{\sim}{x}, t) \in \mathbb{R}^d$ such that

$$\frac{\partial \rho}{\partial t} + \underset{\sim}{\nabla}_x \cdot (\underset{\sim}{u}\, \rho) = 0 \qquad \text{in } \Omega \times (0, T], \quad (1.1a)$$

$$\rho(\underset{\sim}{x}, 0) = \rho_0(\underset{\sim}{x}) \qquad \forall \underset{\sim}{x} \in \Omega, \quad (1.1b)$$

$$\frac{\partial (\rho\, \underset{\sim}{u})}{\partial t} + \underset{\sim}{\nabla}_x \cdot (\rho\, \underset{\sim}{u} \otimes \underset{\sim}{u}) - \underset{\sim}{\nabla}_x \cdot \underset{\approx}{S}(\underset{\sim}{u}, \rho) + \underset{\sim}{\nabla}_x\, p(\rho) = \rho\, \underset{\sim}{f} + \underset{\sim}{\nabla}_x \cdot \underset{\approx}{\tau} \qquad \text{in } \Omega \times (0, T], \quad (1.1c)$$

$$\underset{\sim}{u} = \underset{\sim}{0} \qquad \text{on } \partial\Omega \times (0, T], \quad (1.1d)$$

$$(\rho\, \underset{\sim}{u})(\underset{\sim}{x}, 0) = (\rho_0\, \underset{\sim}{u}_0)(\underset{\sim}{x}) \qquad \forall \underset{\sim}{x} \in \Omega. \quad (1.1e)$$

It is assumed that each of the equations above has been written in its nondimensional form; ρ denotes a nondimensional solvent density, $\underset{\sim}{u}$ is a nondimensional solvent velocity, defined as the velocity field scaled by the characteristic flow speed U_0. Here $\underset{\approx}{S}(\underset{\sim}{u}, \rho)$ is the Newtonian part of the viscous stress tensor defined by