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A priori estimates for the free boundary problem of incompressible neo-Hookean elastodynamics

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Abstract

A free boundary problem for the incompressible neo-Hookean elastodynamics is studied in two and three spatial dimensions. The *a priori* estimates in Sobolev norms of solutions with the physical vacuum condition are established through a geometrical point of view of Christodoulou and Lindblad (2000) [3]. Some estimates on the second fundamental form and velocity of the free surface are also obtained.

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Keywords: Incompressible elastodynamics; Free boundary; *A priori* estimates; The second fundamental form

1. Introduction

We are concerned with the motion of neo-Hookean elastic waves in an incompressible material for which the deformation or strain is proportional to the stress. Precisely, we consider the free boundary problem of the following incompressible elastodynamic equations of neo-Hookean elastic materials:

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$$v_t + v \cdot \partial v + \partial p = \operatorname{div}(FF^\top), \tag{1.1a}$$

$$F_t + v \cdot \partial F = \partial v F, \tag{1.1b}$$

$$\operatorname{div} v = 0, \quad \operatorname{div} F^\top = 0, \tag{1.1c}$$

in a set

$$\mathscr{D} = \bigcup_{0 \leqslant t \leqslant T} \{t\} \times \mathscr{D}_t,$$

where $\mathscr{D}_t \subset \mathbb{R}^n$, $n = 2$ or 3, is the domain that the material occupies at time $t \in [0, T]$ for some $T > 0$; where $\partial = (\partial_1, \cdots, \partial_n)$ and div are the usual gradient operator and spatial divergence in the Eulerian coordinates with $\partial_i = \partial/\partial x^i$, respectively; $v(t, x) = (v_1(t, x), \cdots, v_n(t, x))$ is the velocity vector field of the fluid, $p(t, x)$ is the pressure, $F(t, x) = (F_{ij}(t, x))$ is the deformation tensor, $F^\top = (F_{ji})$ denotes the transpose of the $n \times n$ matrix F, $FF^\top$ is the Cauchy–Green tensor in the case of neo-Hookean elastic materials (cf. [9,14]); and the notations $(\partial v)_{ij} = \partial_j v_i$, $(\partial v F)_{ij} = (\partial v F)^{ij} = (\partial v)_{ik} F^{kj} = \partial_k v^i F^{kj}$, $\operatorname{div} v = \partial_i v^i$, $(\operatorname{div} F^\top)^i = \partial_j F^{ji}$ follow the Einstein summation convention: $v^i = \delta^{ij} v_j = v_i$ and $F^{ij} = \delta^{ik}\delta^{jl} F_{kl} = F_{ij}$. The boundary conditions on the free boundary:

$$\partial\mathscr{D} = \bigcup_{0 \leqslant t \leqslant T} \{t\} \times \partial\mathscr{D}_t$$

are prescribed as the following:

$$p = 0 \ \text{ on } \partial\mathscr{D}, \tag{1.2a}$$

$$\mathcal{N} \cdot F^\top = 0 \ \text{ on } \partial\mathscr{D}, \tag{1.2b}$$

$$(\partial_t + v \cdot \partial)|_{\partial\mathscr{D}} \in T(\partial\mathscr{D}), \tag{1.2c}$$

where $\mathcal{N}(t, x)$ is the exterior unit normal to the free surface $\partial\mathscr{D}_t$ and $T(\partial\mathscr{D})$ is the tangential space to $\partial\mathscr{D}$. The boundary condition (1.2a) implies that the pressure p vanishes outside the domain, (1.2b) indicates that the normal component of $F^\top$ (i.e., $\mathcal{N}_k F^{kj}$) vanishes on the boundary, and (1.2c) means that the free boundary moves with the velocity v of the material particles, i.e., $v \cdot \mathcal{N} = \kappa$ on $\partial\mathscr{D}_t$ with κ the normal velocity of $\partial\mathscr{D}_t$.

For a simply connected bounded domain $\mathscr{D}_0 \subset \mathbb{R}^n$ that is homeomorphic to the unit ball, and the initial data $(v_0(x), F_0(x))$ satisfying the constraint (1.1c): $\operatorname{div} v_0 = 0$, $\operatorname{div} F_0^\top = 0$, we shall establish *a priori* estimates for the set $\mathscr{D} \subset [0, T] \times \mathbb{R}^n$ and the vector fields v and F solving (1.1)–(1.2) with the initial conditions:

$$\{x : (0, x) \in \mathscr{D}\} = \mathscr{D}_0, \quad (v, F)|_{t=0} = (v_0(x), F_0(x)) \text{ for } x \in \mathscr{D}_0. \tag{1.3}$$

We will study the free boundary problem (1.1)–(1.3) under the following natural condition (cf. [2–4,7,8,10–13,16–18]):

$$\nabla_{\mathcal{N}} p \leqslant -\varepsilon < 0 \text{ on } \partial\mathscr{D}_t, \tag{1.4}$$