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# Global existence for small data of the viscous Green–Naghdi type equations

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**Abstract**

We consider the Cauchy problem for the Green–Naghdi equations with viscosity, for small initial data. It is well-known that adding a second order dissipative term to a hyperbolic system leads to the existence of global smooth solutions, once the hyperbolic system is symmetrizable and the so-called Kawashima–Shizuta condition is satisfied. In a previous work, we have proved that the Green–Naghdi equations can be written in a symmetric form, using the associated Hamiltonian. This system being dispersive, in the sense that it involves third order derivatives, the symmetric form is based on symmetric differential operators. In this paper, we use this structure for an appropriate change of variable to prove that adding viscosity effects through a second order term leads to global existence of smooth solutions, for small data. We also deduce that constant solutions are asymptotically stable.


*Keywords:* Green–Naghdi equations; Viscosity; Small solutions; Symmetric structure; Energy equality; Global existence

## 1. Introduction

The Green–Naghdi system is a shallow water approximation of the water waves problem which models incompressible flows. The vertical and horizontal speeds are averaged vertically.

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Moreover, vertical acceleration is supposed too small to be considered [13]. In other words, Green–Naghdi equations is one order higher in approximation compared to the Saint-Venant (called also isentropic Euler) system [3]. To obtain the latter system, not only the vertical acceleration but also the vertical speed are neglected. This leads to a hyperbolic system of equations whereas the Green–Naghdi equation is dispersive due to the term $\alpha h^2 \ddot{h}$ defined below. In this work, we focus on the Green–Naghdi type equation with a second order viscosity:

$$\begin{cases} \partial_t h + \partial_x (hu) = 0, \\ \partial_t (hu) + \partial_x (hu^2) + \partial_x (gh^2/2 + \alpha h^2 \ddot{h}) = \mu \partial_x (h \partial_x u) \end{cases} \tag{1}$$

We assume that $h(x,t) > 0$, $\alpha$ and $\mu$ are strictly positive and $g$ is the gravity constant. The unknown $h$ represents the fluid height and $u$ its average horizontal speed. Moreover, the material derivative $\dot{()}$ is defined by $\dot{()} = \partial_t () + u\partial_x ()$.

**Remark 1.1.** Let us note that the $\alpha = 0$ case gives us the Saint-Venant system. We can also learn more about the derivation of the system in [21,1,15] for $(\mu, \alpha) = (0, \frac{1}{3})$, and in [7] for $(\mu, \alpha) = (0, \frac{1}{4})$.

It is worth remarking that (1) admits the following energy equality [11,7],

$$\partial_t E + \partial_x \left(u(E + p)\right) = \mu u \partial_x (h \partial_x u), \tag{2}$$

where

$$E(h,u) = gh^2/2 + hu^2/2 + \alpha h^3 (\partial_x u)^2/2,$$

and

$$p(h,u) = gh^2/2 + \alpha h^2 \ddot{h}.$$

Then, we can check that (1) admits a family of relative energy conservation equalities given by

$$\partial_t E_{h_e,u_e} + \partial_x P_{h_e,u_e} = \mu(u - u_e)\partial_x (h \partial_x u), \tag{3}$$

where

$$E_{h_e,u_e}(h,u) = g(h - h_e)^2/2 + h(u - u_e)^2/2 + \alpha h^3 (\partial_x u)^2/2, \tag{4a}$$

and

$$P_{h_e,u_e}(h,u) = u E_{h_e,u_e}(h,u) + (u - u_e) p(h,u) - \frac{g h_e^2}{2} u. \tag{4b}$$

This family is parametrized by $(h_e, u_e) \in \mathbb{R}^2$ with $h_e > 0$.

**Remark 1.2.** Let us assume that $\alpha = 0$. Then, $E(h,u)$ and $E_{h_e,u_e}(h,u)$ are convex entropies for Saint-Venant system.

The dissipative term $\mu \partial_x (h \partial_x u)$ considered here in the right hand side of (1), is presented in [12] and some other references, as the viscosity for Saint-Venant system. Indeed, Saint-