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Blow-up problems for the heat equation with a local nonlinear Neumann boundary condition

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Abstract

This paper estimates the blow-up time for the heat equation $u_t = \Delta u$ with a local nonlinear Neumann boundary condition: The normal derivative $\partial u/\partial n = u^q$ on Γ_1, one piece of the boundary, while on the rest part of the boundary, $\partial u/\partial n = 0$. The motivation of the study is the partial damage to the insulation on the surface of space shuttles caused by high speed flying subjects. We show the finite time blow-up of the solution and estimate both upper and lower bounds of the blow-up time in terms of the area of Γ_1. In many other work, they need the convexity of the domain Ω and only consider the problem with $\Gamma_1 = \partial\Omega$. In this paper, we remove the convexity condition and only require $\partial\Omega$ to be C^2. In addition, we deal with the local nonlinearity, namely Γ_1 can be just part of $\partial\Omega$.

MSC: 35K60; 35K20; 35K05; 35K08

Keywords: Blow-up time; Heat equation; Local nonlinear; Neumann boundary condition

1. Introduction and notations

In this paper, Ω is assumed to be a bounded open set in $\mathbb{R}^n$ $(n \geq 2)$ with $\partial\Omega \in C^2$, Γ_1 and Γ_2 are two disjoint open subsets of $\partial\Omega$ with $\overline{\Gamma}_1 \cup \overline{\Gamma}_2 = \partial\Omega$, $\widetilde{\Gamma} \triangleq \overline{\Gamma}_1 \cap \overline{\Gamma}_2$ is C^1 when being regarded as $\partial\Gamma_1$ or $\partial\Gamma_2$. We study the heat equation with a local nonlinear Neumann boundary condition:

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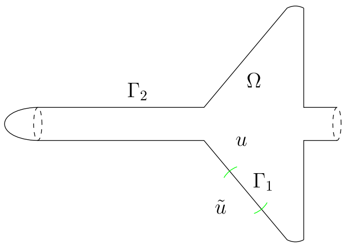

Fig. 1. Model.

$$\begin{cases} u_t(x,t) = \Delta u(x,t) & \text{in} \quad \Omega \times (0,T], \\ \frac{\partial u}{\partial n}(x,t) = u^q(x,t) & \text{on} \quad \Gamma_1 \times (0,T], \\ \frac{\partial u}{\partial n}(x,t) = 0 & \text{on} \quad \Gamma_2 \times (0,T], \\ u(x,0) = u_0(x) & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where $q > 1$, $u_0 \in C^1(\overline{\Omega})$, $u_0(x) \geq 0$ and $u_0(x) \not\equiv 0$. The normal derivative on the boundary is defined as follows: for any $x \in \partial\Omega, 0 < t \leq T$,

$$\frac{\partial u}{\partial n}(x,t) \triangleq \lim_{h \to 0^+} Du(x_h,t) \cdot \overrightarrow{n}(x) \text{ as long as this limit exists,} \tag{1.2}$$

where $\overrightarrow{n}(x)$ denotes the exterior unit normal vector at x and $x_h \triangleq x - h\overrightarrow{n}(x)$ for $x \in \partial\Omega$. $\partial\Omega$ being C^2 ensures that x_h belongs to Ω when h is positive and sufficiently small.

Our work is partially motivated by the Space Shuttle Columbia disaster in 2003. When the space shuttle was launched, a piece of foam broke off from its external tank and struck the left wing causing the insulation there damaged. As a result, the shuttle disintegrated during its reentry to the atmosphere due to the enormous heat generated near the damaged part. Actually, a few previous shuttles also had similar problems but they landed safely. Some engineers suspected that those damages were so small that the shuttles landed before the temperature became huge. Thus, the aim of this paper is to study the relation between the blow-up time of the temperature and the size of the damaged part on the shuttle. We want to support the engineers' speculation from the point view of math. First of all, we start to establish the math model.

In Fig. 1, $\tilde{u}$ represents the outside temperature of the space shuttle and u denotes the inside temperature. When the space shuttle reentered the atmosphere, it compressed the air at a very high speed. During this process, it caused many chemical reactions which produced enormous radiative heat flux. This was the main source of the heat transfer through the broken part on the left wing. In Physics, the radiation heat flux is proportional to the fourth power of the difference between the temperatures. In addition, to simplify the model, we assume $\tilde{u} = F(u)$ is an increasing function of u and treat it as a polynomial, say u^m for some $m > 1$. Thus on the broken part Γ_1, we have

$$\frac{\partial u}{\partial n} \sim (\tilde{u} - u)^4 = (F(u) - u)^4 \sim (u^m - u)^4 \sim u^q,$$

for $q = 4m > 1$. On Γ_2, one has $\frac{\partial u}{\partial n} = 0$, since the insulation there are intact. Inside the space shuttle, we assume it satisfies the heat equation. Thus, the realistic problem is modeled as (1.1).