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A fitness-driven cross-diffusion system from population dynamics as a gradient flow

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Abstract

We consider a fitness-driven model of dispersal of N interacting populations, which was previously studied merely in the case $N = 1$. Based on some optimal transport distance recently introduced, we identify the model as a gradient flow in the metric space of Radon measures. We prove existence of global non-negative weak solutions to the corresponding system of parabolic PDEs, which involves degenerate cross-diffusion. Under some additional hypotheses and using a new multicomponent Poincaré–Beckner functional inequality, we show that the solutions converge exponentially to an ideal free distribution in the long time regime.

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Keywords: Cross-diffusion; Optimal transport; Gradient flow; Ideal free distribution; Entropy–entropy production inequality

1. Introduction

Living organisms tend to form distributional patterns but not to be arranged either uniformly or randomly. This spatial heterogeneity plays a crucial role in ecological theories and their practical applications. It should be taken into account when modeling epidemics, ecological catastro-

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phes, competition, adaptation, maintenance of species diversity, parasitism, population growth and decline, social behavior, and so on [1]. In order to understand the way the species distribute themselves it is important to pay attention to their dispersal strategies.

In this paper we study a system of PDEs for several interacting populations whose dispersal strategy is determined by a local intrinsic characteristic of organisms called *fitness* (cf. [2,3]), essentially the signed difference between available resources and their consumption by the individuals. The fitness manifests itself as a growth rate, and simultaneously affects the dispersal as the species move along its gradient towards the most favorable environment. The equilibrium when the fitnesses of all populations vanish can be referred to as the *ideal free distribution* [4,5], since no net movement of individuals occurs in this case. We are thus going to study the system

$$\partial_t u_i = -\operatorname{div}(u_i \nabla f_i) + u_i f_i, \qquad x \in \Omega, t > 0, \qquad i = 1, \dots, N, \tag{1.1}$$

of N interacting species located in a bounded domain $\Omega \subset \mathbb{R}^d$. For prescribed resources $\mathbf{m} = (m_i(x))$ we assume a generic linear relation between the population densities $\mathbf{u} = (u_i(t,x))$ and their corresponding fitnesses $\mathbf{f} = (f_i(t,x))$:

$$\mathbf{f} = \mathbf{m} - A\mathbf{u}. \tag{1.2}$$

We assume that both the matrix A and the vector $\mathbf{m}$ depend on $x \in \Omega$, thus our model is spatially heterogeneous. Formula (1.2) expresses the idea that the fitness is determined by the difference between the available resources $\mathbf{m}$ and the animals' consumption $A\mathbf{u}$.

Our results can be generalized to the case when (1.1) is replaced by

$$\partial_t u_i = -b_i \operatorname{div}(u_i \nabla f_i) + c_i u_i f_i, \qquad x \in \Omega, t > 0, \qquad i = 1, \dots, N,$$

where the coefficients b_i and c_i are positive numbers. However, to fix the ideas and avoid unnecessary technicalities, hereafter we study only (1.1), (1.2).

The mathematical difficulties which we will face when studying the parabolic system (1.1)–(1.2) come from the fact that it involves both cross-diffusion (for $N > 1$) and degenerate diffusion. In the case of merely one population ($N = 1$), the fitness-driven dispersal model (1.1), (1.2) was suggested in [6,2] and studied in [7,8] (see also [9]). Related fitness-driven two-species models were investigated in [10,11] where one population uses the fitness-driven dispersal strategy and the other diffuses freely or does not move at all. In the case when A is a constant matrix, $\mathbf{m} \equiv 0$, and the second (reaction) term $u_i f_i$ in (1.1) is omitted, system (1.1), (1.2) is equivalent to the degenerate cross diffusion system which was recently analyzed in [12] with an application to seawater intrusion. Another population dynamics model which involves cross-diffusion is the Shigesada, Kawasaki and Teramoto model

$$\partial_t u_i = \Delta\left(u_i\left(d_i + \sum_{i=1}^{N} a_{ij} u_j\right)\right) + u_i\left(\left(c_i - \sum_{i=1}^{N} b_{ij} u_j\right)\right), \; i = 1, \dots, N, \tag{1.3}$$

where the coefficients are non-negative constants. It has been extensively studied (mostly for $N = 2$) from the point of view of well-posedness and long-time behavior (see, e.g., [13–18] and the references therein). Note that the constants d_i in (1.3) are usually assumed to be strictly positive, hence this problem is not as degenerate as our system (1.1), (1.2).