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ScienceDirect
J. Differential Equations 261 (2016) 4368–4423
Journal of Differential Equations
www.elsevier.com/locate/jde

Uniform regularity estimates in homogenization theory of elliptic systems with lower order terms on the Neumann boundary problem ☆

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Received 22 July 2015; revised 18 April 2016
Available online 21 July 2016

Abstract

In this paper, we mainly employed the idea of the previous paper [34] to study the sharp uniform $W^{1,p}$ estimates with $1 < p \leq \infty$ for more general elliptic systems with the Neumann boundary condition on a bounded $C^{1,\eta}$ domain, arising in homogenization theory. Based on the skills developed by Z. Shen in [27] and by T. Suslina in [31,32], we also established the L^2 convergence rates on a bounded $C^{1,1}$ domain and a Lipschitz domain, respectively. Here we found a "rough" version of the first order correctors (see (1.12)), which can unify the proof in [27] and [32]. It allows us to skip the corresponding convergence results on $\mathbb{R}^d$ that are the preconditions in [31,32]. Our results can be regarded as an extension of [23] developed by C. Kenig, F. Lin, Z. Shen, as well as of [32] investigated by T. Suslina.

Keywords: Elliptic systems; Homogenization; Uniform regularity estimates; Neumann boundary problem; Convergence rates

1. Introduction and main results

M. Avellaneda and F. Lin developed the compactness methods in [3,4] to study uniform regularity estimates for Dirichlet problems in homogenization theory in the end of 1980s. For the Neumann boundary value problem, it is not until [23] established by C. Kenig, F. Lin and Z. Shen

☆ The research was supported by the National Natural Science Foundation of China (Grant No. 11471147).
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http://dx.doi.org/10.1016/j.jde.2016.06.027
0022-0396/

in 2013 that there was no significant progress on this topic. Recently, a new method has been introduced in [2,27] by S. Armstrong and Z. Shen to arrive at the sharp regularity estimates, uniformly down to the microscopic scale, without smoothness assumptions, for Dirichlet and Neumann problems in periodic or non-periodic settings. Meanwhile T. Suslina [31,32] derived the sharp $O(\varepsilon)$ convergence rate in $L^2(\Omega)$ for elliptic systems with either Dirichlet or Neumann boundary conditions in a $C^{1,1}$ domain.

Inspired by these papers, we originally investigated some uniform regularity estimates for the elliptic operator with rapidly oscillating potentials that is

$$\mathfrak{L}_\varepsilon(u_\varepsilon) = -\Delta u_\varepsilon + \frac{1}{\varepsilon}\mathcal{W}(x/\varepsilon)u_\varepsilon + \lambda u_\varepsilon = F \quad \text{in } \Omega,$$

where $\mathcal{W}$ is referred to as the rapidly oscillating potential term (see [5, p. 93]). As we have shown in [34], the operator $\mathfrak{L}_\varepsilon$ is only a special case of $\mathcal{L}_\varepsilon$, and therefore indicates that our results are not very trivial as it seems to be.

Returning to this paper, neither the well-known compactness methods nor the new developed technique is rigidly used. Instead we try to make full use of the previous work in [23]. On account of these results, we mainly establish the uniform $W^{1,p}$ estimates ($1 < p \leq \infty$), as well as the L^2 convergence rates for more general elliptic systems with the Neumann boundary condition in homogenization theory. More precisely, we consider the following operators depending on parameter $\varepsilon > 0$,

$$\mathcal{L}_\varepsilon = -\mathrm{div}\big[A(x/\varepsilon)\nabla + V(x/\varepsilon)\big] + B(x/\varepsilon)\nabla + c(x/\varepsilon) + \lambda I$$

where $\lambda \geq 0$ is a constant, and $I = (e^{\alpha\beta})$ is an identity matrix.

Let $d \geq 3$, $m \geq 1$, and $1 \leq i, j \leq d$ and $1 \leq \alpha, \beta \leq m$. Suppose that $A = (a_{ij}^{\alpha\beta})$, $V = (V_i^{\alpha\beta})$, $B = (B_i^{\alpha\beta})$, $c = (c^{\alpha\beta})$ are real measurable functions, satisfying the following conditions:

- the uniform ellipticity condition

$$\mu|\xi|^2 \leq a_{ij}^{\alpha\beta}(y)\xi_i^\alpha\xi_j^\beta \leq \mu^{-1}|\xi|^2, \quad \text{for } y \in \mathbb{R}^d, \text{ and } \xi = (\xi_i^\alpha) \in \mathbb{R}^{md}, \text{ where } \mu > 0; \tag{1.1}$$

  (The summation convention for repeated indices is used throughout.)
- the periodicity condition

$$\begin{aligned} &A(y+z) = A(y), \quad V(y+z) = V(y), \quad B(y+z) = B(y), \quad c(y+z) = c(y), \\ &\text{for } y \in \mathbb{R}^d \text{ and } z \in \mathbb{Z}^d; \end{aligned} \tag{1.2}$$

- the boundedness condition

$$\max\left\{\|V\|_{L^\infty(\mathbb{R}^d)}, \ \|B\|_{L^\infty(\mathbb{R}^d)}, \ \|c\|_{L^\infty(\mathbb{R}^d)}\right\} \leq \kappa_1, \qquad \text{where } \kappa_1 > 0; \tag{1.3}$$

- the regularity condition

$$\max\left\{\|A\|_{C^{0,\tau}(\mathbb{R}^d)}, \ \|V\|_{C^{0,\tau}(\mathbb{R}^d)}\right\} \leq \kappa_2, \qquad \text{where } \tau \in (0,1) \text{ and } \kappa_2 > 0. \tag{1.4}$$