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Singularly perturbed control systems with noncompact fast variable ☆

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Abstract

We deal with a singularly perturbed optimal control problem with slow and fast variable depending on a parameter ε. We study the asymptotics, as ε goes to 0, of the corresponding value functions, and show convergence, in the sense of weak semilimits, to sub and supersolution of a suitable limit equation containing the effective Hamiltonian.

The novelty of our contribution is that no compactness condition is assumed on the fast variable. This generalization requires, in order to perform the asymptotic procedure, an accurate qualitative analysis of some auxiliary equations posed on the space of fast variable. The task is accomplished using some tools of Weak KAM theory, and in particular the notion of Aubry set.

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1. Introduction

We study a singularly perturbed optimal control problem with a slow variable, say x, and a fast one, denoted by y, with dynamics depending on a parameter ε devoted to become infinitesimal.

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We are interested in the asymptotics, as ε goes to 0, of the corresponding value functions V^ε, depending on slow, fast variable and time, in view of proving convergence, in the sense of weak semilimits, to some functions independent of y, related to a limit control problem where y does not appear any more, at least as state variable.

More precisely, we exploit that the V^ε are solutions, in the viscosity sense, to a time-dependent Hamilton–Jacobi–Bellman equation of the form

$$u_t^\varepsilon + H\left(x, y, D_x u^\varepsilon, \frac{D_y u^\varepsilon}{\varepsilon}\right) = 0$$

and show that the upper/lower weak semilimit is sub/supersolution to a limit equation

$$u_t + \overline{H}(x, Du) = 0$$

containing the so-called effective Hamiltonian $\overline{H}$, obtained via a canonical procedure we describe below from the Hamiltonian of the approximating equations. We also show that initial conditions, i.e. terminal costs, are transferred, with suitable adaptations, to the limit. See Theorems 4.3, 4.4, which are the main results of the paper.

We tackle the subject through a PDE approach first proposed in this context by Alvarez–Bardi, see [1,2] and the booklet [3], in turn inspired by techniques developed in the framework of homogenization of Hamilton–Jacobi equations by Lions–Papanicolau–Varadhan and Evans, see [18,12,13]. The singular perturbation can be actually viewed as a relative homogenization of slow with respect to fast variable. In the original formulation, homogenization was obtained assuming periodicity in the underlying space plus coercivity of the Hamiltonian in the momentum variable.

Alvarez–Bardi keep periodicity in y, but do without coercivity, and assume instead bounded time controllability in the fast variable. A condition of this kind is indeed unavoidable, otherwise it cannot be expected to get rid of y at the limit, or even to get any limit. Another noncoercive homogenization problem, arising from turbulent combustion models, has been recently investigated with similar techniques in [19].

The novelty of our contribution is that we remove any compactness condition on the fast variable, and this requires major adaptations in the perturbed test function method, which is the core of the asymptotic procedure. We further comment on it later on.

Following a more classical control-theoretic approach, namely directly working on the trajectory of the dynamics, Arstein–Gaitsgory, see [7] and [5,6], have studied a similar model replacing in a sense periodicity by a coercivity condition in the cost, and allowing y to vary in the whole of $\mathbb{R}^M$, for some dimension M. Besides proving convergence, they also provide a thorough description of the limit control problem, in terms of occupational measures, see [6]. This is clearly a relevant aspect of the topic, but we do not treat it here.

Our aim is to recover their results adapting Alvarez–Bardi techniques. We assume, as in [7] and [5], coercivity of running cost, see **(H4)**, and a controllability condition, see **(H3)**, stronger than the one used in [1–3] and implying, see Lemma 2.9, coercivity of the corresponding Hamiltonian, at least in the fast variable. We do believe that our methods can also work under bounded time controllability, and so without any coercivity on H, but this requires more work, and the details have still to be fully checked and written down.

The focus of our analysis is on the associate cell problem, namely the one-parameter family of stationary equations, posed in the space of fast variable, obtained by freezing in H slow variable and momentum, say at a value (x_0, p_0). Its role, at least in the periodic case, is twofold: