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The Hartman–Grobman theorem for semilinear hyperbolic evolution equations

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Abstract

The famous Hartman–Grobman theorem for ordinary differential equations is extended to abstract semilinear hyperbolic evolution equations in Banach spaces by means of simple direct proof. It is also shown that the linearising map is Hölder continuous. Several applications to abstract and specific damped wave equations are given, to demonstrate the strength of our results.

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1. Introduction

The statement of the classical *Hartman–Grobman theorem*, also known as *linearisation theorem*, for ordinary differential equations is as follows. Nearby the *hyperbolic equilibrium* $x_* = 0$, the behaviour of the nonlinear system $\dot{x} = Ax + r(x)$, $x(0) = x_0$, where $r \in C^1(\mathbb{R}^d)$,

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$r(x) = o(|x|)$, is by means of a bi-continuous bijective coordinate transformation completely described by the linear problem $\dot{y} = Ay$. Here *hyperbolic* means that the matrix $A \in \mathbb{R}^{d\times d}$ has no purely imaginary eigenvalues.

One should be aware of the fact that even in the finite-dimensional case, the bi-continuous bijective map $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ which linearises the nonlinear problem according to

$$x(t; \Phi(x_0)) = \Phi(e^{At}x_0), \quad t \in \mathbb{R},\ x_0 \in \mathbb{R}^d,$$

is in general not unique, and also in general is not in C^1; see Chicone [3] for more details. This fact is responsible for several analytical difficulties in the proof of the Hartman–Grobman theorem, even in the finite-dimensional case.

The aim of this paper is to transfer this result from the theory of ordinary differential equations to semilinear *hyperbolic* evolution equations. This is a non-trivial task, as in the finite-dimensional proofs at some point topological arguments are used, like Brouwers fixed point theorem. Such arguments are not available in the infinite dimensional case, in general, in particular not in the hyperbolic case.

To be more precise, let A be the generator of a C_0-group e^{At} on the Banach space X and $r\colon X \to X$ be Lipschitz continuous. In analogy to the finite-dimensional case, we examine the semilinear and associated linear initial value problems

$$\begin{cases} \partial_t v = Av + r(v), \\ v(t_0) = v_0 \in X, \end{cases} \quad \text{and} \quad \begin{cases} \partial_t u = Au, \\ u(t_0) = u_0 \in X. \end{cases}$$

The linear problem is well-understood within the framework of semigroup theory. It is well-known that the semilinear equation admits a unique global mild solution $v(t; v_0)$. So the question is for a link between the solution $u(t; u_0) = e^{At}u_0$ of the linear system and the solution $v(t; v_0)$ of the semilinear initial value problem. More precisely, is there an analogue of the Hartman–Grobman theorem for semilinear evolution equations? It is the goal of this paper to prove such a statement in a global formulation as well as a local version, for semilinear hyperbolic evolution equations. Note that except for the finite dimensional case, a C_0-group is never compact, so there is no intrinsic compactness available. Therefore, the usual finite-dimensional proofs cannot be extended to the infinite dimensional case.

Our basic assumptions are that the operator A generates a C_0-group in the Banach space X which admits a dichotomy. For the global version of our main result, we require that the nonlinearity r is bounded, $r(0) = 0$, and globally Lipschitz continuous with sufficiently small Lipschitz constant $|r|_{Lip}$. The main result reads as follows.

Theorem 1.1. *Let X be a Banach space, A the generator of the C_0-group e^{At} on X which admits an exponential dichotomy. Suppose that $r\colon X \to X$ is bounded and Lipschitz continuous, with $r(0) = 0$ and sufficiently small Lipschitz constant.*

Then there is a homeomorphism $\Phi\colon X \to X$ such that

$$\Phi\left(e^{At}x\right) = v(t; \Phi(x)), \quad t \in \mathbb{R},\ x \in X, \tag{1.1}$$

where $v(t; x)$ denotes the mild solution of