Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 261 (2016) 5306–5323

Journal of Differential Equations

www.elsevier.com/locate/jde

# Three-region inequalities for the second order elliptic equation with discontinuous coefficients and size estimate

E. Francini [a], C.-L. Lin [b], S. Vessella [a], J.-N. Wang [c,*]

[a] Universitá di Firenze, Italy
[b] National Cheng Kung University, Taiwan
[c] National Taiwan University, Taiwan

Received 21 September 2015; revised 27 July 2016

Available online 20 August 2016

**Abstract**

In this paper, we would like to derive a quantitative uniqueness estimate, the three-region inequality, for the second order elliptic equation with jump discontinuous coefficients. The derivation of the inequality relies on the Carleman estimate proved in our previous work [5]. We then apply the three-region inequality to study the size estimate problem with one boundary measurement.


*Keywords:* Carleman estimate; Three-region inequalities; Discontinuous coefficients; Size estimate

## 1. Introduction

In this work we aim to study the size estimate problem with one measurement when the background conductivity has jump interfaces. A typical application of this study is to estimate the size of a cancerous tumor inside an organ by the electric impedance tomography (EIT). In this case, considering discontinuous medium is typical, for instance, the conductivities of heart, liver,

* Corresponding author.
*E-mail addresses:* francini@math.unifi.it (E. Francini), cllin2@mail.ncku.edu.tw (C.-L. Lin), sergio.vessella@dmd.unifi.it (S. Vessella), jnwang@ntu.edu.tw (J.-N. Wang).

http://dx.doi.org/10.1016/j.jde.2016.08.002
0022-0396/

intestines are 0.70 (s/m), 0.10 (s/m), 0.03 (s/m), respectively. Previous works on this problem assumed that the conductivity of the studied body is Lipschitz continuous, see, for example, [3, 4]. The first result on the size estimate problem with a discontinuous background conductivity was given in [18], where only the two dimensional case was considered. In this paper, we will study the problem in dimension $n \geq 2$.

The main ingredients of our method are quantitative uniqueness estimates for

$$\mathrm{div}(A\nabla u) = 0 \quad \Omega \subset \mathbb{R}^n. \tag{1.1}$$

Those estimates are well-known when $A$ is Lipschitz continuous. The derivation of the estimates is based on the Carleman estimate or the frequency function method. For $n = 2$ and $A \in L^\infty$, quantitative uniqueness estimates are obtained via the connection between (1.1) and quasiregular mappings. This is the method used in [18]. For $n \geq 3$, the connection with quasiregular mappings is not true. Hence we return to the old method – the Carleman estimate, to derive quantitative uniqueness estimates when $A$ is discontinuous. Precisely, when $A$ has a $C^{1,1}$ interface and is Lipschitz away from the interface, a Carleman estimate was obtained in [5] (see [11–13] for related results). Here we will derive three-region inequalities using this Carleman estimate. The three-region inequality provides us a way to propagate "smallness" across the interface (see also [12] for similar estimates). Relying on the three-region inequality, we then derive bounds of the size of an inclusion with one boundary measurement. For other results on the size estimate, we mention [1] for the isotropic elasticity, [15–17] for the isotropic/anisotropic thin plate, [7,6] for the shallow shell.

## 2. The Carleman estimate

In this section, we would like to describe the Carleman estimate derived in [5]. We first denote $H_\pm = \chi_{\mathbb{R}^n_\pm}$ where $\mathbb{R}^n_\pm = \{(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R} : y \gtrless 0\}$ and $\chi_{\mathbb{R}^n_\pm}$ is the characteristic function of $\mathbb{R}^n_\pm$. Let $u_\pm \in C^\infty(\mathbb{R}^n)$ and define

$$u = H_+u_+ + H_-u_- = \sum_\pm H_\pm u_\pm,$$

hereafter, $\sum_\pm a_\pm = a_+ + a_-$, and

$$\mathcal{L}(x, y, \partial)u := \sum_\pm H_\pm \mathrm{div}_{x,y}(A_\pm(x, y)\nabla_{x,y}u_\pm), \tag{2.1}$$

where

$$A_\pm(x, y) = \{a^\pm_{ij}(x, y)\}^n_{i,j=1}, \quad x \in \mathbb{R}^{n-1}, y \in \mathbb{R} \tag{2.2}$$

is a Lipschitz symmetric matrix-valued function satisfying, for given constants $\lambda_0 \in (0, 1]$, $M_0 > 0$,

$$\lambda_0|z|^2 \leq A_\pm(x, y)z \cdot z \leq \lambda_0^{-1}|z|^2, \ \forall(x, y) \in \mathbb{R}^n, \ \forall z \in \mathbb{R}^n \tag{2.3}$$

and