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Limiting classification on linearized eigenvalue problems for 1-dimensional Allen–Cahn equation II — Asymptotic profiles of eigenfunctions

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Abstract

This paper is a continuation of a previous paper by the authors. We are interested in the asymptotic behavior of eigenpairs on one dimensional linearized eigenvalue problem for Allen–Cahn equations as the diffusion coefficient tends to zero. We obtain the asymptotic profiles of all eigenfunctions by using the asymptotic formulas of corresponding eigenvalues, which have been obtained in the previous paper. Our results lead us to the concept of the classification of limiting eigenfunctions. In the case of Allen–Cahn equation it is provided by three special eigenfunctions, which correspond to the solutions of rescaled spectral problems on the whole line.

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1. Introduction and main results

Let us consider the nonlinear boundary value problem

$$\begin{cases} \varepsilon^2 u_{xx}(x) + f(u(x)) = 0 & \text{in } (0,1), \\ u_x(0) = u_x(1) = 0, \end{cases} \tag{1.1}$$

and its linearized eigenvalue problem associated with $u(x)$

$$\begin{cases} \varepsilon^2 \varphi_{xx}(x) + f_u(u(x))\varphi(x) + \lambda\varphi(x) = 0 & \text{in } (0,1), \\ \varphi_x(0) = \varphi_x(1) = 0, \end{cases} \tag{1.2}$$

where ε is a positive parameter and f is smooth enough. The problem (1.1) is the stationary problem of a reaction diffusion equation, which generates a gradient system on a suitable function space. We say u is *n-mode solution* of (1.1) if u is nontrivial solution of (1.1) and it admits exactly n zeros $z_\ell = z_\ell^n$ $(\ell = 1, \dots, n)$. Suppose that f is a *balanced bistable nonlinearity*: f has exactly three zeros $u_- < 0 < u_+$ and they satisfy

$$f_u(0) > 0, \quad f_u(u_\pm) < 0, \quad F(u_+) = F(u_-), \tag{1.3}$$

where

$$F(u) := \int_0^u f(s)ds.$$

For simplicity, we also assume f is odd. It is well known that for arbitrarily $n \in \mathbf{N}$, (1.1) admits exactly two n-mode solutions $\pm u_{n,\varepsilon}(x)$ with $u_{n,\varepsilon}(0) > 0$ when ε is small enough. For more details and theoretical backgrounds, see Chafee–Infante [3] and Henry [5].

The linearized eigenvalue problem (1.2) for $u = u_{n,\varepsilon}$ is rewritten as

$$\begin{cases} \varepsilon^2 \varphi_{xx}(x) + f_u(u_{n,\varepsilon}(x))\varphi(x) + \lambda\varphi(x) = 0 & \text{in } (0,1), \\ \varphi_x(0) = \varphi_x(1) = 0. \end{cases} \tag{1.4}$$

For $j \in \mathbf{N} \cup \{0\}$ we denote by $\lambda_j = \lambda_j^{n,\varepsilon}$ and $\varphi_j(x) = \varphi_j^{n,\varepsilon}(x)$, the $(j+1)$-th eigenvalue and the corresponding eigenfunction. It is also well known that $\lambda_0^{n,\varepsilon} < \cdots < \lambda_{n-1}^{n,\varepsilon} < 0 < \lambda_n^{n,\varepsilon} < \cdots < +\infty$; $u_{n,\varepsilon}$ is unstable. Therefore, information on $\varphi_j^{n,\varepsilon}$ with $0 \le j < n$, which correspond to the unstable eigenvalues, plays an important role in understanding the dynamical theory of the reaction–diffusion equations (see e.g., Brunovský–Fiedler [1]).

Let us consider a situation that $n \in \mathbf{N}$ is arbitrarily fixed, and that ε is sufficiently small. In this case $u_{n,\varepsilon}$ admits *transition layers* in a neighborhood of its zeros, and correspondingly, the potential $f_u(u_{n,\varepsilon})$ of (1.4) admits the localized patterns called *spikes* (see Figs. 1 and 2). Then, such a formation of localized patterns is also expected for $\varphi_{j,\varepsilon}^n$. Spikes of $\varphi_{j,\varepsilon}^n$ with $0 \le j < n$, have been discussed by Carr–Pego [2] and Fusco–Hale [4]. Furthermore, an interesting conjecture is shown by E. Yanagida: