Available online at www.sciencedirect.com

ScienceDirect

J. Differential Equations 261 (2016) 5590–5608

Journal of Differential Equations

www.elsevier.com/locate/jde

On the locally self-similar solution of the surface quasi-geostrophic equation with decaying or non-decaying profiles

Liutang Xue

School of Mathematical Sciences, Beijing Normal University and Laboratory of Mathematics and Complex Systems, Ministry of Education, Beijing 100875, PR China

Received 4 April 2016

Available online 20 August 2016

Abstract

Motivated by the numerical simulation and the study on several 1D models, we consider the locally self-similar singular solutions for the surface quasi-geostrophic equation with decaying or non-decaying blowup profiles. Based on a suitable local L^p-inequality in terms of the profile and the bootstrapping method, we show some exclusion results and derive the asymptotic behavior of the possible blowup profiles.

MSC: 76B03; 35Q31; 35Q35; 35Q86

Keywords: Locally self-similar solutions; Surface quasi-geostrophic equation; Blowup profiles; Exclusion results

1. Introduction

In this paper we address the Cauchy problem of the surface quasi-geostrophic (abbr. SQG) equation

E-mail address: xuelt@bnu.edu.cn.

http://dx.doi.org/10.1016/j.jde.2016.08.012
0022-0396/

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta = 0, \\ u = (u_1, u_2) = (-\mathcal{R}_2 \theta, \mathcal{R}_1 \theta), \\ \theta|_{t=0} = \theta_0, \end{cases} \tag{1.1}$$

where $(x,t) \in \mathbb{R}^2 \times \mathbb{R}^+$, $\mathcal{R}_i = \partial_i |D|^{-1}$ $(i = 1, 2)$ is the usual Riesz transform, $\theta : \mathbb{R}^2 \to \mathbb{R}$ is a scalar field understood as temperature or density field, and $u : \mathbb{R}^2 \to \mathbb{R}^2$ is the velocity field of $\mathbb{R}^2$. The SQG equation arises from the geostrophic study of the highly rotating flow (cf. [17]) and is viewed as a 2D simple model sharing much formal analogy with the 3D Euler equations (cf. [8]). It is known for some time that the SQG equation associated with smooth initial data generates a local smooth solution, e.g., for $\theta_0 \in H^s(\mathbb{R}^2)$, $s > 2$, there exists a unique solution $\theta \in C([0, T[; H^s(\mathbb{R}^2))$ with some $T > 0$ and u is expressed as

$$u(x,t) = \text{p.v.} \int_{\mathbb{R}^2} K^{\perp}(x - z) \theta(z, t) \mathrm{d}z, \tag{1.2}$$

with $K^{\perp}(z) := \frac{1}{2\pi} \frac{z^{\perp}}{|z|^3}$, $\forall z \neq 0$; moreover, if additionally $\theta_0 \in L^p(\mathbb{R}^2)$ with $p \in [1, \infty]$, the solution also satisfies $\theta \in L^\infty([0, T[; L^p(\mathbb{R}^2))$ with $\|\theta\|_{L^\infty([0,T[;L^p(\mathbb{R}^2))} \le \|\theta_0\|_{L^p(\mathbb{R}^2)}$ (e.g. [14, Proposition 6.2]). In recent years there have been intense mathematical works on the SQG equation and its dissipative cases (one can see [2] for a long list of references), but so far the fundamental problem: whether the local smooth solutions remain regular forever or develop blowup singularity at finite time, remains completely open.

Some numerical simulation was also made to understand the issue of finite-time blowup. It was suggested by [15,18,13], and very recently by [19,20] that the finite-time singularity formulation for the SQG equation is possible to happen, via a self-similar cascade of filament instabilities of geometrically decreasing spatial and temporal scales. Other past numerical studies [8,9,16] mainly focused on the flow of a closing saddle geometry, and through a much higher resolution than [9,16], Scott in [19] pointed out that the self-similar type filament instability mentioned above was also potentially important in this scenario.

Motivated by the numerical work, we here mainly focus on the self-similar singular solution for the SQG equation (1.1), i.e., the solution of the form

$$\theta(x,t) = \frac{1}{(T-t)^{\frac{\alpha}{1+\alpha}}} \Theta\left(\frac{x - x_0}{(T-t)^{\frac{1}{1+\alpha}}} \right), \quad \forall (x,t) \in D \times]0, T[, \tag{1.3}$$

where $\alpha > -1$, $x_0 \in D$, $D \subset \mathbb{R}^2$ is the blowup region, $T > 0$ is the finite blowup time, $\Theta(\cdot)$ is the stationary profiles of θ, and the solution θ is regular enough on $(\mathbb{R}^2 \setminus D) \times]0, T[$. If $D = \mathbb{R}^2$, the self-similar solution (1.3) is referred to as the globally self-similar solution; while if $D \subsetneq \mathbb{R}^2$, (1.3) is called the locally self-similar solution. For the globally self-similar solution (1.3), the velocity field u is also globally self-similar satisfying that

$$u(x,t) = \frac{1}{(T-t)^{\frac{\alpha}{1+\alpha}}} U\left(\frac{x - x_0}{(T-t)^{\frac{1}{1+\alpha}}} \right), \quad \forall (x,t) \in \mathbb{R}^2 \times]0, T[, \tag{1.4}$$

with $U(\cdot)$ the velocity profile. In terms of (Θ, U), we formally get