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Diffusion phenomena for the wave equation with space-dependent damping in an exterior domain

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Abstract

In this paper, we consider the asymptotic behavior of solutions to the wave equation with space-dependent damping in an exterior domain. We prove that when the damping is effective, the solution is approximated by that of the corresponding heat equation as time tends to infinity. Our proof is based on semigroup estimates for the corresponding heat equation and weighted energy estimates for the damped wave equation. The optimality of the decay late for solutions is also established.

MSC: 35L20; 35C06; 47B25; 35B40

Keywords: Damped wave equation; Diffusion phenomena; Friedrichs extensions; Semigroup estimates; Weighted energy estimates

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ be an exterior domain with smooth boundary. We consider the initial-boundary value problem to the wave equation with space-dependent damping

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$$\begin{cases} u_{tt} - \Delta u + a(x)u_t = 0, & x \in \Omega,\ t > 0, \\ u(x,t) = 0, & x \in \partial\Omega,\ t > 0, \\ (u, u_t)(x, 0) = (u_0, u_1)(x), & x \in \Omega. \end{cases} \tag{1.1}$$

Here $u = u(x,t)$ is a real-valued unknown function. The coefficient of the damping term $a(x)$ is a radially symmetric function on the whole space $\mathbb{R}^N$ satisfying $a \in C^2(\mathbb{R}^N)$ and

$$a(x) = a_0|x|^{-\alpha} + o(|x|^{-\alpha}) \quad \text{as } |x| \to \infty \tag{a0}$$

with some $a_0 > 0$ and $\alpha \in [0, 1)$, and then we may assume $0 \notin \overline{\Omega}$ without loss of generality. The initial data (u_0, u_1) belong to $[H^3(\Omega) \cap H_0^1(\Omega)] \times [H^2(\Omega) \cap H_0^1(\Omega)]$ with the compatibility condition of second order and satisfy $\operatorname{supp}(u_0, u_1) \subset \{x \in \Omega; |x| < R_0\}$ with some $R_0 > 0$. Here we recall that for a nonnegative integer k, with the assumption $a \in C^{\max(k,2)}$, the initial data $(u_0, u_1) \in H^{k+1} \times H^k$ satisfy the compatibility condition of order k if $u_p = 0$ on $\partial\Omega$ for $p = 0, 1, \dots, k$, where u_p are successively defined by $u_p = \Delta u_{p-2} - a(x)u_{p-1}$ $(p = 2, \dots, k)$. Then, it is known that (1.1) admits a unique solution

$$u \in \bigcap_{i=0}^{k+1} C^i([0,\infty); H^{k+1-i}(\Omega))$$

(see Ikawa [1, Theorem 2]).

Also, we consider the initial-boundary value problem to the corresponding heat equation

$$\begin{cases} v_t - a(x)^{-1}\Delta v = 0, & x \in \Omega, t > 0, \\ v(x,t) = 0, & x \in \partial\Omega, t > 0, \\ v(x,0) = v_0(x), & x \in \Omega. \end{cases} \tag{1.2}$$

Our aim is to prove that the asymptotic profile of the solution to (1.1) is given by a solution of (1.2) as time tends to infinity. Namely, the solution of the damped wave equation (1.1) has the diffusion phenomena.

The diffusive structure of the damped wave equation has been studied for a long time. Matsumura [8] proved L^p–L^q estimates of solutions in the case $\Omega = \mathbb{R}^N$ and $a(x) \equiv 1$. On the one hand, Mochizuki [13] considered the case where $\Omega = \mathbb{R}^N$ and the coefficient $a = a(x,t)$ satisfies $0 \le a(x,t) \le C(1+|x|)^{-\alpha}$ with $\alpha > 1$, and proved that in general the energy of the solution does not decay to zero. Moreover, for some initial data, the solution approaches to a solution to the wave equation without damping in the energy sense. Mochizuki and Nakazawa [14] generalized it to the case of exterior domains with star-shaped complement. Matsuyama [10] further extended it to more general domains by adding the assumption of the positivity of $a(x,t)$ around $\partial\Omega$.

On the other hand, when the coefficient a satisfies $a(x) \ge C(1+|x|)^{-\alpha}$ with some $\alpha \in [0, 1)$, Matsumura [9] and Uesaka [24] showed that the energy of the solution decays to zero. When $\Omega = \mathbb{R}^N$ and the coefficient $a(x)$ is radially symmetric and satisfies (a0), Todorova and Yordanov [23] introduced a suitable weight function of the form $t^{-m}e^{\psi}$, which originates from [22] and [3], and proved an almost optimal energy estimate

$$\int_{\mathbb{R}^N} (|u_t|^2 + |\nabla u|^2)dx \le C(1+t)^{-\frac{N-\alpha}{2-\alpha}-1+\varepsilon}\|(u_0,u_1)\|_{H^1\times L^2}^2.$$