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Global bifurcation of positive solutions for a class of superlinear elliptic systems

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Abstract

We consider a system of semilinear equations of the form

$$\left.\begin{array}{lll} -\Delta u = \lambda f(v) & \text{in} & \Omega; \\ -\Delta v = \lambda g(u) & \text{in} & \Omega; \\ u = 0 = v & \text{on} & \partial\Omega, \end{array}\right\}$$

where $\lambda \in \mathbb{R}$ is the bifurcation parameter, $\Omega \subset \mathbb{R}^N$; $N \geq 2$ is a bounded domain with smooth boundary $\partial\Omega$. The nonlinearities $f, g : \mathbb{R} \to (0, +\infty)$ are nondecreasing continuous functions that have superlinear growth at infinity. We use bifurcation theory, combined with an approximation scheme, to establish the existence of an unbounded branch of positive solutions, emanating from the origin, which is bounded in positive λ-direction.

If in addition, Ω is convex and $f,\ g \in C^1$ satisfy certain subcriticality condition, we show that the branch must bifurcate from infinity at $\lambda = 0$.

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Keywords: Elliptic systems; Positive solutions; Superlinear growth; Bifurcation from infinity

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1. Introduction

We study an elliptic system of the form

$$\left.\begin{array}{lll} -\Delta u = \lambda f(v) & \text{in} & \Omega; \\ -\Delta v = \lambda g(u) & \text{in} & \Omega; \\ u = 0 = v & \text{on} & \partial\Omega, \end{array}\right\} \tag{1.1}$$

where $\lambda \in \mathbb{R}$ is the bifurcation parameter and $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with $C^{2,\eta}$-boundary $\partial\Omega$ for some $\eta \in (0,1)$. The nonlinearities f and g satisfy the following assumptions:

(H1) $f, g : \mathbb{R} \to (0, +\infty)$ are nondecreasing continuous functions; and
(H2) $\lim_{s\to+\infty} \frac{f(s)}{s} = +\infty = \lim_{s\to+\infty} \frac{g(s)}{s}$.

The goal of this manuscript is to use bifurcation theory to study positive solutions of (1.1). Thus we begin by setting up function spaces and defining terminologies before stating our result.

Let $E \stackrel{\text{def}}{=} \left[W_0^{1,2}(\Omega) \cap W^{2,r}(\Omega)\right]^2$ and $X \stackrel{\text{def}}{=} \left[L^r(\Omega)\right]^2$ be Banach spaces endowed with norms $\|(w_1, w_2)\|_E \stackrel{\text{def}}{=} \|w_1\|_{W^{2,r}(\Omega)} + \|w_2\|_{W^{2,r}(\Omega)}$ and $\|(w_1, w_2)\|_X \stackrel{\text{def}}{=} \|w_1\|_{L^r(\Omega)} + \|w_2\|_{L^r(\Omega)}$, respectively for $r > N$. By a *solution* of (1.1) we mean $(\lambda, (u, v)) \in \mathbb{R} \times E$ which solves (1.1) in the strong sense, that is, $(u, v) \in W^{2,r}(\Omega) \times W^{2,r}(\Omega)$ and $(\lambda, (u, v))$ satisfies (1.1) almost everywhere in Ω. If $(\lambda, (u, v)) \in \mathbb{R} \times E$ is a solution, then $(u, v) \in E$ is called a solution of (1.1) corresponding to λ. These solutions are called *strong solutions* of (1.1). If $u > 0$ and $v > 0$ almost everywhere in Ω, then we say that $(\lambda, (u, v))$ is a positive solution of (1.1). By the maximum principle, it is easy to see that $(\lambda, (u, v)) \in \mathscr{S}$ is positive whenever $\lambda > 0$, where $\mathscr{S} \stackrel{\text{def}}{=} \{(\lambda, (u, v)) \in \mathbb{R} \times E : (\lambda, (u, v)) \text{ solution of (1.1)}\}$. By a *continuum* of solutions of (1.1) we mean a subset $\mathscr{K} \subset \mathscr{S}$ which is closed and connected. By a *component* of solution set $\mathscr{S}$ we mean a continuum which is maximal with respect to inclusion ordering. We say that $\lambda_\infty \in \mathbb{R}$ is a *bifurcation point* from infinity if the solution set $\mathscr{S}$ contains a sequence $(\lambda_n, (u_n, v_n))$ such that $\lambda_n \to \lambda_\infty$ and $\|(u_n, v_n)\|_E \to +\infty$ as $n \to +\infty$. We say that a continuum $\mathscr{K}$ bifurcates from infinity at $\lambda_\infty \in \mathbb{R}$ if there exists a sequence of solutions $(\lambda_n, (u_n, v_n)) \in \mathscr{K}$ such that $\lambda_n \to \lambda_\infty$ and $\|(u_n, v_n)\|_E \to +\infty$ as $n \to +\infty$. We extend the use of these definitions to all systems throughout the paper.

Let $\mu_1 > 0$ be the principal eigenvalue of

$$\left.\begin{array}{lll} -\Delta\varphi = \lambda\varphi & \text{in} & \Omega; \\ \varphi = 0 & \text{on} & \partial\Omega, \end{array}\right\} \tag{1.2}$$

and $\varphi_1 \in W_0^{1,2}(\Omega)$ be the corresponding eigenfunction. Without loss of generality, we normalize the eigenfunction such that $\varphi_1 > 0$ in Ω. By the standard regularity argument, $\varphi_1 \in W_0^{1,2}(\Omega) \cap W^{2,r}(\Omega)$ for any $r > N$, and it is a strong solution of (1.2). Then it is well known that $\frac{\partial\varphi_1}{\partial\vec{n}} < 0$ on $\partial\Omega$, where $\vec{n}$ is the outward unit normal on $\partial\Omega$.

We first state the following nonexistence result which holds under weaker assumptions than (H1)–(H2).