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An abstract theorem on the existence of periodic motions of non-autonomous Lagrange systems

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Abstract

In this paper, we established an abstract theorem on the existence of periodic solutions for Lagrange systems under Tonelli framework (strictly convex and superlinear, completeness of phase flow) which is based on the principle of least action in geometry and dynamics. An example of the application of the abstract theorem is given and the existence of periodic solutions for a class of time-periodic Lagrange systems will be proved.

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1. Introduction

In this paper, we investigate periodic solutions with prescribed integer period of Lagrange system

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$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}}(t, x(t), \dot{x}(t)) = \frac{\partial L}{\partial x}(t, x(t), \dot{x}(t)), \quad \text{in } \mathbb{R}^N \tag{1.1}$$

given by the Tonelli Lagrangian $L : \mathbb{R}/\mathbb{Z} \times T\mathbb{R}^N \to \mathbb{R}$ with global Euler–Lagrange flow. Here and in the sequel the upper dot denotes a derivative with respect to time.

Equation (1.1) describes all models of Lagrangian mechanical systems, which are defined by a configuration space and a function on its tangent bundle, namely, the Lagrangian function. The Lagrangian in classical mechanics is given by

$$L(t, x, \dot{x}) = T - U = \frac{1}{2}\sum_{i=0}^{N} m_i \dot{x}_i^2 - U(x),$$

where T, U denotes the kinetic energy and the potential energy, respectively. Under the Legendre transformation

$$x = x, \quad p = \frac{\partial L}{\partial \dot{x}}(t, x, \dot{x}),$$

the Lagrange system is equivalent to the Hamiltonian system

$$\dot{x} = \frac{\partial H}{\partial p}(t, x, p), \quad \dot{p} = -\frac{\partial H}{\partial x}(t, x, p), \quad (x, p) \in T^*\mathbb{R}^N,$$

where the Hamiltonian is given by $H(t, x, p) = p \cdot \dot{x} - L(t, x, \dot{x})$ with the variable p defined implicitly by $p = \partial L/\partial \dot{x}(t, x, \dot{x})$.

In case that the Lagrangian is given by

$$L(t, x, \dot{x}) = \bar{L}(t, \dot{x}) - V(t, x),$$

where $V(t, x)$ is T-periodic with respect to t, the existence of periodic solutions for the corresponding Euler–Lagrange equation has been studied by many authors such as some of cases for superlinear V_x [1,2] and sublinear V_x [3–5]. It is interesting that the method used in [2] is based on the dual variational method for the periodic problem

$$\begin{cases} \dfrac{\mathrm{d}p(t)}{\mathrm{d}t} + V_x(t, x) = 0, \quad \text{in } \mathbb{R}, \\ p(t) = \bar{L}_{\dot{x}}(t, \dot{x}), \\ x(0) = x(T), \ p(0) = p(T) \end{cases} \tag{1.2}$$

according to the idea discovered by Clarke, where $\bar{L}(t, \cdot)$ and $V(t, x)$ are convex and differentiable in the second variable and satisfy the quadratic type of power function. The results were extended by Nowakowski and Rogowski in [6], where they improved the method to relax the convexity assumption on the function V.

Moreover, we remark that when the configuration space is a closed manifold, the standard symplectic m-torus, the problem on periodic solutions of (1.1) is related to Conley conjecture [7], which was established by Conley and Zehnder for generic Hamiltonian and by Hingston