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Spatially localized solutions of the Hammerstein equation with sigmoid type of nonlinearity

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Abstract

We study the existence of fixed points to a parameterized Hammerstein operator $\mathcal{H}_\beta$, $\beta \in (0, \infty]$, with sigmoid type of nonlinearity. The parameter $\beta < \infty$ indicates the steepness of the slope of a nonlinear smooth sigmoid function and the limit case $\beta = \infty$ corresponds to a discontinuous unit step function. We prove that spatially localized solutions to the fixed point problem for large β exist and can be approximated by the fixed points of $\mathcal{H}_\infty$. These results are of a high importance in biological applications where one often approximates the smooth sigmoid by discontinuous unit step function. Moreover, in order to achieve even better approximation than a solution of the limit problem, we employ the iterative method that has several advantages compared to other existing methods. For example, this method can be used to construct non-isolated homoclinic orbit of a Hamiltonian system of equations. We illustrate the results and advantages of the numerical method for stationary versions of the FitzHugh–Nagumo reaction–diffusion equation and a neural field model.

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1. Introduction

We study the existence of solutions to the fixed point problem

$$u = \mathcal{H}_\beta u, \quad (\mathcal{H}_\beta u)(x) := \int_{\mathbb{R}} \omega(x-y) f_\beta(u(y))dy. \tag{1.1}$$

Here $\mathcal{H}_\beta$ is the parameterized Hammerstein operator with $\beta \in (0, \infty]$, $\omega(x)$ is symmetric, and $f_\beta(u) : \mathbb{R} \to [0, 1]$ is a smooth function of sigmoid shape that approaches (in some way which we specify later) the unit step function $f_\infty = \chi_{[h,\infty)}$ for some $h > 0$ as $\beta \to \infty$. Examples of this type of functions are

$$f_\beta(u) = S(\beta(u-h)), \quad S(u) := \frac{1}{1+\exp(-|u|)}, \tag{1.2}$$

and

$$f_\beta(u) = S(\beta(u-h)), \quad S(u, p) := \frac{u^p}{u^p+1}\chi_{[0,\infty)}(u), \ p > 0, \tag{1.3}$$

see Fig. 1.

This problem arises in several biological applications, e.g., in studying the existence of steady state solutions to a neural field model and the FitzHugh–Nagumo equation. We give examples of these models in Section 2.

When the function f_β is such that $f_\beta(u) = 0$ for all $u < h$, i.e., $\text{supp}(f_\beta) \subset [h, \infty)$, all solutions to (1.1) can be divided into two categories: (i) localized solutions and (ii) non-localized solutions (e.g. periodic, quasi-periodic). Here we study the solutions of the first class which we define in detail in Section 4.2.

For the limit case $\beta = \infty$, one often can construct localized solutions analytically, see e.g. [1] and Chapter 3 in [2]. However, the case $\beta = \infty$ is only a simplification of a more realistic model where f_β, $0 < \beta < \infty$ is a steep yet smooth function. Analytical tools do not work in the latter case, and the existence of solutions for the case of $\beta < \infty$ and their continuous dependents on β as $\beta \to 0$, is often only conjectured from numerical simulations, see e.g. [3–5].

The main challenge of a proper justification of the transition between the cases $\beta < \infty$ and $\beta = \infty$, is discontinuity of f_∞ which leads to discontinuity of the corresponding integral operator in any standard functional space. To avoid this difficulty we suggest to exploit a spatial structure of localized solutions for $\beta = \infty$ and construct functional spaces such that the operator $\mathcal{H}_\infty$ is not only continuous but Fréchet differentiable and the Implicit Function Theorem can be used. That is, we show that under the assumption that localized solution of (1.1) for $\beta = \infty$ exists and satisfies some properties, solutions to (1.1) for large $\beta < \infty$ exist, converge to a solution of